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# ON THE MATHEMATICS OF SIMPLE CORRELATION

C. D. Smith

*Introduction.* The method of correlation is one of the large fields in applied mathematics. Discussions appear in books and periodicals covering a wide range of apparently different objectives. In some cases the proper meaning and use of results is not clear. Workers who are not mathematically trained may find difficulty in planning a problem so that proper interpretations may be assigned to measures of correlation. If we begin with the original definition of correlation the mathematics involved is elementary and an exposition of the mathematical developments should lead to a clearer understanding of the place of the measures of correlation in applied problems. In this paper we first give a general statement of the problem and then the formulas for measuring correlation are developed beginning with Karl Pearson's original definition.

*A Statement of the Problem.* Variables in nature, science, commerce, and other fields take a set of values which are determined by the combined influence of many factors. If we assume two variables, $X_1$ and $X_2$, so that a value of one variable determines a corresponding value of the other, we say they have a functional relation which may be given by $X_1 = f(X_2)$. Substitution of a value of $X_2$ in the equation gives the corresponding value of $X_1$. As a rule we do not find that a functional relation exists between variables. In some cases it is true that the variables are independent in the sense that a value of one variable has no influence upon values of the other. For example, one variable may represent weights of members of a foot ball team and the other may represent the burning time of light bulbs. In another case one may assume that some common property may cause $X_1$ to vary in a manner similar to that of $X_2$. For example, let $X_1$ represent the height of a group of sons and $X_2$ the heights of their fathers. It is well known that a group of fathers of approximately average height produce sons which average near the height of all sons. But if a set of fathers average ten percent above average it does not follow that their sons average ten percent above the average of all sons. We believe that the degree of similarity is due to heredity and many factors other than the father will influence height. There is no equation that will give the height of a son by substituting the height of his father. Other cases arise where variation seems to run in opposite directions. The problem of correlation arises when we select what seems to be the dominating factor of common influence

and use that as basis for assigning the pairs of values to $X_1$ and $X_2$.

By correlation we seek to measure the strength of the common influences between two variables by using sample values of each variable. To do this we arrange the observations in pairs using the criterion which we believe to represent common influence while other factors are randomized. For example, let the problem be to measure joint ability of students in basic english and in mathematics. Comparable tests are given to a group of students. The answers are rated by the same person and each student's grades constitute a pair, $(X_1, X_2)$. Assume that any degree of similarity between grades is due to joint ability and any other influences are randomized. Even now we cannot say that correspondence is produced entirely by the basis for pairing. We can say that a high degree of similarity increases the probability that the result indicates joint ability in the two subjects. The method of correlation is designed to measure the strength of common influences between two variables. By calculating a measure of simple correlation we hope to obtain good estimates of one variable which correspond to observed values of the other.

*The Pearsonian Coefficient of Correlation.* The principles which lead to Pearson's method originated with the work of Frances Galton. Darwin's theory of evolution relative to animal life lead Galton to studies of animals and plants by collecting and classifying samples. Results indicated cases of similarity between samples. By plotting the pairs of values as points on rectangular co-ordinates he found cases where values of one variable seemed to correspond to median values of the other variable. In some cases the median points followed the general direction of a straight line and in other cases they seemed to follow some type of curve. He drew what he called lines of regression to indicate the path which median points seemed to follow most closely. Galton founded the Biometric Laboratory in London where the results of his studies were kept. Realizing that mathematical training would be required to study measures of relation, he invited his nephew, Karl Pearson, to serve as research assistant in the laboratory to study the possibilities of measuring the strength of correspondence by mathematical calculations. In 1891 Pearson published his method of calculating a coefficient of correlation to be used as a measure of the strength of correspondence between pairs of values of two variables.

He began by considering the arithmetic means of the arrays of a variable $X_1$ which correspond to assigned values of the other variable $X_2$. A straight line is fitted to the values of $X_1$ by the method of least squares. Let $x_1 = X_1 - \overline{X}_1$ and $x_2 = X_2 - \overline{X}_2$. Fit the linear equation $x_1 = a + bx_2$. Values of $a$ and $b$ are calculated from sample data so that the sum of the squares of the deviations of observed values from the corresponding points

on the line shall be a minimum. The necessary conditions are;

$$\sum x_1 = \sum a + b\sum x_2$$

$$\sum x_1 x_2 = a\sum x_2 + b\sum x_2^2$$

Since $x_1$ and $x_2$ are deviations from their respective means we have $\Sigma x_1 = \Sigma x_2 = 0$. Substitution in the two equations give $a = 0$ and $b = \frac{\Sigma x_1 x_2}{\Sigma x_2^2}$. Now the equation $x_1 = a + bx_2$ becomes

$$x_1 = \frac{\Sigma x_1 x_2}{\Sigma x_2^2} x_2.$$

Next calculate the variance $(\sigma_e^2)$ of values on the line and also the variance $(\sigma_1^2)$ of the observed values of $x_1$. The coefficient of correlation is defined as $r = \pm\frac{\sigma_e}{\sigma_1}$, where $\sigma_e$ is calculated from the linear estimates of $x_1$. When the equation $x_2 = a + bx_1$ is fitted by least squares the result is

$$x_2 = \frac{\Sigma x_1 x_2}{\Sigma x_1^2} x_1, \qquad \text{and } r = \pm\frac{\sigma_e}{\sigma_2},$$

where $\sigma_e$ is calculated from estimates of $x_2$. It turns out that the two expressions for $r$ have the same value. We prove the truth of the statement and write the two regression equations in terms of $r$ and the standard deviations.

Beginning with $x_1 = \frac{\Sigma x_1 x_2}{\Sigma x_2^2} x_2$ write

$$r^2 = \frac{\sigma_e^2}{\sigma_1^2} = \frac{\frac{1}{N}\left(\frac{\Sigma x_1 x_2}{\Sigma x_2^2}\right)^2 \Sigma x_2^2}{\frac{1}{N}(\Sigma x_1^2)} = \frac{\frac{1}{N}(\Sigma x_1 x_2)^2}{\frac{1}{N}\Sigma x_1^2 \Sigma x_2^2} = \frac{\frac{1}{N^2}(\Sigma x_1 x_2)^2}{\frac{1}{N}\Sigma x_1^2 \cdot \frac{1}{N}\Sigma x_2^2}.$$

The square root gives, $r = \frac{\frac{1}{N}\Sigma x_1 x_2}{\sigma_1 \sigma_2}$. Next begin with $x_2 = \frac{\Sigma x_1 x_2}{\Sigma x_1^2} x_1$ and write

$$r^2 = \frac{\sigma_e^2}{\sigma_2^2} = \frac{\frac{1}{N}\left(\frac{\Sigma x_1 x_2}{\Sigma x_1^2}\right)^2 \Sigma x_1^2}{\frac{1}{N}\Sigma x_2^2} = \frac{\frac{1}{N}(\Sigma x_1 x_2)^2}{\frac{1}{N}\Sigma x_1^2 \Sigma x_2^2} = \frac{\frac{1}{N^2}(\Sigma x_1 x_2)^2}{\frac{1}{N}\Sigma x_1^2 \cdot \frac{1}{N}\Sigma x_2^2}.$$

The square root gives, $r = \dfrac{\frac{1}{N}\Sigma x_1 x_2}{\sigma_1 \sigma_2}$.

To write the formula for $r$ in terms of observed values begin with

$$\frac{1}{N}\Sigma x_1 x_2 = \frac{1}{N}\Sigma(X_1 - \overline{X}_1)(X_2 - \overline{X}_2) = \frac{1}{N}\Sigma(X_1 X_2 - \overline{X}_1 X_2 - X_1 \overline{X}_2 + \overline{X}_1 \overline{X}_2)$$

$$= \frac{1}{N}\Sigma X_1 X_2 - \overline{X}_1 \cdot \frac{1}{N}\Sigma X_2 - \overline{X}_2 \cdot \frac{1}{N}\Sigma X_1 + \overline{X}_1 \overline{X}_2 = \frac{1}{N}\Sigma X_1 X_2 - \overline{X}_1 \overline{X}_2.$$

Substitution of this result for the numerator gives

$$r = \frac{\frac{1}{N}\Sigma X_1 X_2 - \overline{X}_1 \overline{X}_2}{\sigma_1 \sigma_2}.$$

Begin with $x_1 = \dfrac{\Sigma x_1 x_2}{\Sigma x_2^2} x_2$, and write

$$\frac{\frac{1}{N}\Sigma x_1 x_2}{\frac{1}{N}\Sigma x_2^2} \cdot x_2 = \frac{\frac{1}{N}\Sigma x_1 x_2}{\sigma_2^2} \cdot \frac{\sigma_2}{\sigma_1} \cdot \frac{\sigma_1}{\sigma_2} \cdot x_2 = \frac{\frac{1}{N}\Sigma x_1 x_2}{\sigma_1 \sigma_2} \cdot \frac{\sigma_1}{\sigma_2} x_2 = r\frac{\sigma_1}{\sigma_2} x_2.$$

The result is $x_1 = r\dfrac{\sigma_1}{\sigma_2} x_2$.

Begin with $x_2 = \dfrac{\Sigma x_1 x_2}{\Sigma x_1^2} x_1$ and a similar reduction on the right hand side gives $x_2 = r\dfrac{\sigma_2}{\sigma_1} x_1$.

Obviously the two regression equations may be written in terms of observed values. Remembering that $x_1 = X_1 - \overline{X}_1$ and $x_2 = X_2 - \overline{X}_2$ we have directly by substitution

$$X_1 - \overline{X}_1 = r\frac{\sigma_1}{\sigma_2}(X_2 - \overline{X}_2) \text{ and}$$

$$X_2 - \overline{X}_2 = r\frac{\sigma_2}{\sigma_1}(X_1 - \overline{X}_1).$$

To apply the method of correlation we calculate the averages, the standard deviations, and $r$ from the sample values. By substituting the value of one variable on the right side of the proper equation we calculate the corresponding value of the other. If we represent the variance of the means of arrays of $X_1$ by $\sigma_m^2$, the Correlation Ratio is defined as $\eta = \pm\dfrac{\sigma_m}{\sigma_1}$. A

design for correlation charts is given in one of the following sections of the paper for convenience in arranging the necessary calculations in cases of large samples.

*Some Important Properties or $r$ and $\eta$.* If a constant value $c$ is added to each value of $X_1$ the correlation between $X_1 + c$ and $X_2$ is the same as that for $X_1$ and $X_2$. If each value of $X_1$ is multiplied by a constant $c$ the correlation is not changed. Since $c$ may be negative or a fraction we may subtract a constant or divide by a constant without changing the value of $r$. The same statements apply to changes in $X_2$. To prove all cases it is sufficient to prove the cases for $X_1 + c$ and for $cX_1$.

First begin with the formula $r = \dfrac{\frac{1}{N}\Sigma x_1 x_2}{\sigma_1 \sigma_2}$, and prove the formula $r = \dfrac{\frac{1}{N}\Sigma x x_2}{\sigma_x \sigma_2}$, where $x = X_1 + c - \overline{X_1 + c}$.

$$\overline{X_1 + c} = \frac{1}{N}\Sigma(X_1 + c) = \frac{1}{N}\Sigma X_1 + \frac{1}{N}\Sigma c = \overline{X}_1 + c. \text{ Then } x = X_1 - \overline{X}_1 = x_1.$$

The result is that $\sigma_x = \sigma_1$ and the value of the coefficient $r$ is not changed.

Next change $X_1$ by substituting $cX_1$ and let $x_c$ be a deviation from the average of $cX_1$. Let $\sigma_c$ be the standard deviation of $cX_1$. Then write,

$$x_c = cX_1 - \overline{cX_1} = cX_1 - \frac{1}{N}\Sigma cX_1 = cX_1 - c\overline{X}_1 = c(X_1 - \overline{X}_1) = cx_1.$$

Also

$$\sigma^2_{cx_1} = \frac{1}{N}\Sigma(cx_1)^2 = c^2\frac{1}{N}\Sigma x_1^2 = c^2\sigma_1^2, \text{ and } \sigma_c = c\sigma_1.$$

Then

$$r = \frac{\frac{1}{N}\Sigma cx_1 x_2}{\sigma_c \sigma_2} = \frac{c\frac{1}{N}\Sigma x_1 x_2}{c\sigma_1 \sigma_2} = \frac{\frac{1}{N}\Sigma x_1 x_2}{\sigma_1 \sigma_2},$$

and the value of $r$ is not changed.

Limiting values of $r$ and $\eta$ are given by the relation,

$$-1 \leqq -\eta \leqq -r \leqq 0 \leqq r \leqq \eta \leqq 1.$$

To prove the relation we first let $d$ represent the distance of a point $X_1$ from the corresponding point on the regression line, and write $d = x_1 - r\frac{\sigma_1}{\sigma_2}x_2$. To calculate the variance of points about the regression line write

$$S_1^2 = \frac{1}{N}\Sigma d^2 = \frac{1}{N}\Sigma\left(x_1 - r\frac{\sigma_1}{\sigma_2}x_2\right)^2$$

$$= \frac{1}{N}\left(\Sigma x_1^2 - 2r\frac{\sigma_1}{\sigma_2}\Sigma x_1 x_2 + r^2\frac{\sigma_1^2}{\sigma_2^2}\Sigma x_2^2\right)$$

$$= \sigma_1^2 - 2r\frac{\sigma_1}{\sigma_2}(r\sigma_1\sigma_2) + r^2\frac{\sigma_1^2}{\sigma_2^2}\cdot\sigma_2^2$$

$$= \sigma_1^2 - 2r^2\sigma_1^2 + r^2\sigma_1^2$$

$$= \sigma_1^2(1-r^2).$$

If all points of the arrays should fall on the regression line we would have $S_1^2 = 0$, and $r = \pm 1$, which indicates perfect correlation. If the regression line is horizontal and passes through the mean of $X_1$ we will have $S_1^2 = \sigma_1^2$ and $r = 0$, so that we have no correlation between $X_1$ and $X_2$. Using the definition $r^2 = \dfrac{\sigma_e^2}{\sigma_1^2}$ in the equation $S_1^2 = \sigma_1^2(1-r^2)$ gives

$$S_1^2 = \sigma_1^2\left(1 - \frac{\sigma_e^2}{\sigma_1^2}\right) = \sigma_1^2 - \sigma_e^2.$$

Write the result in the form $S_1^2 + \sigma_e^2 = \sigma_1^2$ and we see that the total variance $\sigma_1^2$ is the sum of the two variances, $S_1^2$ of observed points about the corresponding points on the line, and $\sigma_e^2$ of points on the line about the sample mean.

If $S_m^2$ is the variance of points about the means of arrays of $X_1$ relative to assigned values of $X_2$ and $\sigma_m^2$ is the variance of means of arrays about the general mean, Pearson proved the relation, $S_m^2 + \sigma_m^2 = \sigma_1^2$.

Begin by letting $x - m$ represent the distance from a point in any given array to the mean $m$ of that array. We may then write,

$$x = (x-m) + m,$$

where $x$ is the distance from a point to the general sample mean. By squaring both sides we have,

$$x^2 = (x-m)^2 + 2(x-m)m + m^2.$$

Writing the sums on both sides gives the equation,

$$\Sigma x^2 = \Sigma(x-m)^2 + 2\Sigma(x-m)m + \Sigma m^2.$$

Divide the product term by the number $N$ of points in the array and sum the results. We then have,

$$\frac{1}{N}\Sigma(x-m)m = \frac{1}{N}\Sigma xm - \frac{1}{N}\Sigma m^2 = m\frac{1}{N}\Sigma x - m^2 = m^2 - m^2 = 0.$$

Since the product vanishes for the sum in a given array it will vanish in the sum of all the arrays and the final sums give the relation,

$$\frac{1}{N}\Sigma x^2 = \frac{1}{N}\Sigma(x-m)^2 + \frac{1}{N}\Sigma m^2.$$

Noting that each term represents a variance we have the result,

$$\sigma_1^2 = S_m^2 + \sigma_m^2,$$

which is true regardless of the manner in which values of $X_1$ are assigned to the arrays. When $X_1$ is arrayed according to assigned values of $X_2$, Pearson defined the correlation ratio as $\eta = \pm\frac{\sigma_m}{\sigma_1}$. We now write

$$S_m^2 = \sigma_1^2\left(1 - \frac{\sigma_m^2}{\sigma_1^2}\right) = \sigma_1^2(1-\eta^2).$$

Since the variance $S_m^2$ about the means of arrays is the minimum variance, it follows that $\eta$ is the maximum measure of correlation and we have the numerical relation $\eta \geqq r$. Also since the relation, $S_m^2 = \sigma_1^2(1-\eta^2)$, indicates $\eta \leqq 1$, numerically the general expression indicating relative boundary values for $r$ and $\eta$ is written,

$$-1 \leqq -\eta \leqq -r \leqq 0 \leqq r \leqq \eta \leqq 1.$$

R. A. Fisher used the relation $\sigma_1^2 = \sigma_m^2 + S_m^2$ as a basis for his theory of the analysis of variance. The relation holds when the sums are divided by the corresponding number of degrees of freedom which gives better estimates of variance in cases of small samples. Small sub sets occur in many cases such as values of a variable occurring at different times, or by production of different varieties of beans grown on land of similar quality. The ratio $F = \frac{\sigma_m^2}{S_m^2}$ was used by Fisher to test the nature of the differences between sub sets of samples. Assuming that $S_m^2$ represents random variation within sub sets the $F$-test determines the estimate of probability that the differences between sub sets is greater than differences due to random variation. Tables have been published which give the probabilities

associated with certain values of $F$.

The discussions in this section indicate the very close relationship between Fisher's theory of variance and Pearson's theory of correlation. In fact, both theories depend upon the same mathematical relation, $\sigma_1^2 = \sigma_m^2 + S_m^2$. Pearson divides $\sigma_m^2$ by $\sigma_1^2$ to determine $\eta^2$, and Fisher divides $\sigma_m^2$ by $S_m^2$ to determine $F$.

*Crathorne's Formula for the Correlation Ratio.* In 1912 a formula for calculating the correlation ratio from sample values was published by A. R. Crathorne. The formula may be developed in the following manner.

Begin with the definition $\eta_{12}^2 = \dfrac{\sigma_m^2}{\sigma_1^2}$. Let the mean of an array of $X_1$ be given by $m = \dfrac{S_{x_1}}{f_{x_2}}$, where $S_{x_1}$ is the sum of values in the array of $X_1$, and $f_{x_2}$ is the number of points in the array. Then $m - \overline{X}_1 = \dfrac{S_{x_1}}{f_{x_2}} - \overline{X}_1$. Starting with the definition we have,

$$\eta_{12}^2 = \frac{\sigma_m^2}{\sigma_1^2} = \frac{1}{\sigma_1^2}\cdot\frac{1}{N}\Sigma f_{x_2}(m - \overline{X}_1)^2 = \frac{1}{\sigma_1^2}\cdot\frac{1}{N}\Sigma f_{x_2}\left(\frac{S_{x_1}}{f_{x_2}} - \overline{X}_1\right)^2. \text{ Then,}$$

$$\begin{aligned}
\frac{1}{N}\Sigma f_{x_2}\left(\frac{S_{x_1}}{f_{x_2}} - \overline{X}_1\right)^2 &= \frac{1}{N}\Sigma\left(\frac{S_{x_1}^2}{f_{x_2}} - 2\overline{X}_1 S_{x_1} + f_{x_2}\overline{X}_1^2\right) \\
&= \frac{1}{N}\Sigma\frac{S_{x_1}^2}{f_{x_2}} - 2\overline{X}_1\cdot\frac{1}{N}\Sigma S_{x_1} + \overline{X}_1^2\cdot\frac{1}{N}\Sigma f_{x_2} \\
&= \frac{1}{N}\Sigma\frac{S_{x_1}^2}{f_{x_2}} - 2\overline{X}_1^2 + \overline{X}_1^2 \\
&= \frac{1}{N}\Sigma\frac{S_{x_1}^2}{f_{x_2}} - \overline{X}_1^2.
\end{aligned}$$

This result in the right hand side of the formula gives,

$$\eta_{12}^2 = \frac{1}{\sigma_1^2}\left(\frac{1}{N}\Sigma\frac{S_{x_1}^2}{f_{x_2}} - \overline{X}_1^2\right).$$

Begin with the definition, $\eta_{21}^2 = \frac{\sigma_{m_2}^2}{\sigma_2^2}$, and a similar development gives,

$$\eta_{21}^2 = \frac{1}{\sigma_2^2}\left(\frac{1}{N}\Sigma \frac{S_{x_2}^2}{f_{x_1}} - \overline{X}_2^2\right).$$

The formulas are calculated directly from the correlation table and the square roots of the results give the two correlation ratios. A convenient pattern for correlation frequency charts is given in a later section showing the arrangement of calculations for use in the formulas for the correlation coefficient and the correlation ratios.

*The Frequency Chart for Correlation of Large Samples.*

Correlation by using each pair of sample values may become a very long process in cases involving large samples. To shorten the process we divide the range of each variable into classes of equal width placing the classes in order with one variable range horizontal and the other vertical. The pairs of classes determine cells which represent class values of the variables. Each pair of values is plotted as a point falling in the corresponding cell of the table. Assume that the mid point of a cell represents the class numbers and assign consecutive numbers to the classes on each range using any convenient class as origin. The number of points falling in a cell gives the cell frequency. By adding the frequencies in the columns and also in the rows of the chart we obtain the marginal frequency distributions of the classes. The correlation between the classes of values is given by correlating the class numbers. To calculate the correlation between classes let $f$ represent the class frequency so that the sample number $N = \Sigma f$, and $\Sigma X_1 X_2 = \Sigma f X_1 X_2$. In a former section we have proved that $r$ is not changed by measuring the variable from different points as origin. The following proof shows that $r$ is not changed by correlating the class numbers measured from arbitrary origins.

Assign to $X_1$ classes of equal width $i_1$ and to $X_2$ classes of equal width $i_2$. Represent class values of $X_1$ by $v$ and class values of $X_2$ by $u$, where $v$ and $u$ represent midvalues of the classes and are counted as consecutive whole numbers from any convenient class as origin. We have $X_1 = i_1 v$, $X_2 = i_2 u$, $\sigma_1 = i_1\sigma_v$, $\sigma_2 = i_2\sigma_u$, $\overline{X}_1 = i_1\overline{v}$, and $\overline{X}_2 = i_2\overline{u}$. Begin with the formula

$$r = \frac{\frac{1}{N}\Sigma f X_1 X_2 - \overline{X}_1\overline{X}_2}{\sigma_1\sigma_2}.$$

Substitution of the equivalent values gives

$$r = \frac{\frac{1}{N}\Sigma f(i_1 v)(i_2 u) - (i_1\overline{v})(i_2\overline{u})}{(i_1\sigma_v)(i_2\sigma_u)}.$$

The factor $i_1 i_2$ appears in numerator and denominator so that cancellation gives

$$r = \frac{\frac{1}{N}\Sigma fuv - \overline{u}\overline{v}}{\sigma_u\sigma_v}.$$

The corresponding formulas for correlation ratios give

$$\eta_{12} = \frac{1}{\sigma_v}\left(\frac{1}{N}\Sigma\frac{S_v^2}{f_u} - \overline{v}^2\right)^{½},$$

and

$$\eta_{21} = \frac{1}{\sigma_u}\left(\frac{1}{N}\Sigma\frac{S_u^2}{f_v} - \overline{u}^2\right)^{½}.$$

A convenient design may be arranged, as follows, for calculating $r$ and the correlation ratios.

Correlation Frequency Chart for Pairs $(X_1X_2)$

| $X_1 \backslash X_2$ | mid class values | | | | $f_v$ | $v$ | $vf_v$ | $v^2f_v$ | $\Sigma uf$ $=S_u$ | $vS_u$ | $S_u^2$ | $\frac{S_u^2}{f_v}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| mid class values | | | | $f$ | | | | | | | | |
| | | $f$ | $f$ | | | | | | | | | |
| | | $f$ | $f$ | | | | | | | | | |
| | $f$ | $f$ | | | | | | | | | | |
| $f_u$ | | | | | $\Sigma f$ | | $\Sigma vf_v$ | $\Sigma v^2f_v$ | | $\Sigma vS_u$ | | $\Sigma\frac{S_u^2}{f_v}$ |
| $u$ | | | | | | | | | | | | |
| $uf_u$ | | | | | $\Sigma uf_u$ | | | | | | | |
| $u^2f_u$ | | | | | $\Sigma u^2f_u$ | | | | | | | |
| $\Sigma vf$ $=S_v$ | | | | | | | | | | | | |
| $uS_v$ | | | | | $\Sigma uS_v$ | | | | | | | |
| $S_v^2$ | | | | | | | | | | | | |
| $\frac{S_v^2}{f_u}$ | | | | | $\Sigma\frac{S_v^2}{f_u}$ | | | | | | | |

$\Sigma fuv$

The marginal sums are used to calculate, $\overline{u}$, $\overline{v}$, $\sigma_u$, $\sigma_v$, $r$, $\eta_{12}$, and $\eta_{21}$.

Pearson showed that $r$ may be used to measure correlation when the sample number $N$ is large and $N(\eta^2-r^2) \leqq 11.37$. The classes used as origins for counting $u$ and $v$ are given in the chart to illustrate the method of counting. Any class as origin for counting $u$ or $v$ will give the same values for $r$ and the correlation ratios. The discussion up to this point constitutes the basis for applying the method of linear correlation.

*Non Linear Correlation.* In cases when it is obvious that the points on a correlation chart seem to follow a curve better than a straight line the correlation may be improved by fitting the equation of a suitable curve. As a rule the equations representing curves are far more difficult to fit by the method of least squares than was the case of the straight line.

To illustrate the method we will use the relatively simple case of parabolas represented by, $x_1 = a + bx_2 + cx_2^2$. Here $x_1$ and $x_2$ are deviations from their respective means. The relations which satisfy the criterion of least squares may be written as follows;

$$\Sigma x_1 = \Sigma a + b\Sigma x_2 + c\Sigma x_2^2.$$

$$\Sigma x_1 x_2 = a\Sigma x_2 + b\Sigma x_2^2 + c\Sigma x_2^3.$$

$$\Sigma x_1 x_2^2 = a\Sigma x_2^2 + b\Sigma x_2^3 + c\Sigma x_2^4.$$

We know that $\Sigma x_1 = 0$, $\Sigma x_2 = 0$, and $\Sigma x_2^3 = 0$. Substitutions in the three equations give

$$0 = Na + c\Sigma x_2^2.$$

$$\Sigma x_1 x_2 = b\Sigma x_2^2.$$

$$\Sigma x_1 x_2^2 = a\Sigma x_2^2 + c\Sigma x_2^4.$$

By calculating the indicated sums from the sample we may substitute in the three equations and solve the set simultaneously for the values of $a$, $b$, and $c$. Substitution in $x_1 = a + bx_2 + cx_2^2$ gives the parabola of best fit. One may estimate a value of $x_1$ by substituting the corresponding value of $x_2$ and solving the equation. If we represent the variance of the estimates of $x_1$ by $\sigma_e^2$, the non linear coefficient of correlation is defined as, $\rho_{12} = \pm \frac{\sigma_e}{\sigma_1}$.

If we begin with the equation, $x_2 = a + bx_1 + cx_1^2$, and fit to values of $x_2$ by least squares, the result may be used to estimate $x_2$ by substituting the corresponding value of $x_1$ in the equation. In this case the correlation is defined as, $\rho_{21} = \pm \frac{\sigma_e}{\sigma_2}$. The value of $r$ was found to be the same when

calculated from each regression line, but the two non linear coefficients are not necessarily the same. For this reason it is convenient to write $\rho_{12}$ when estimating $x_1$ from $x_2$, and $\rho_{21}$ when estimating $x_2$ from $x_1$. The relations involving corresponding values of $r$, $\rho$, and $\eta$, are given by the boundary conditions, $r \leqq \rho \leqq \eta$. Because of the difficulties which arise in calculation of non linear correlations it is customary to use $r$ except in cases when a very simple curve will greatly improve the results.

*Some Problems Related to Simple Correlation.* The method of simple correlation has been extended to problems involving more than two variables. The more important extensions include, multiple correlations, partial correlations, multiple-partial correlations, and vector correlations, where the results are functions of the simple correlations involved. Other special problems which require particular formulas for measuring relation include correspondence between sets of ranks, and sets of broad categories which appear in contengency tables. Kendall developed a method called $t$-correlation which measures the relation in cases of ordered ranks. The method has been extended to include more than two sets of ranks.

Other modifications of the method of correlation have been used to compare sets of non measured characteristics. For example, a sample of individuals may be classified in income groups : High income, Above average, Average, Below average, and Low income. They may also be classified by type of job : Executive, Foreman, Skilled labor, and Common labor. One procedure is to plot the pairs for each individual on a chart and determine the marginal distributions for each set. Each distribution is fitted to a normal range of values and the normal values are correlated by a suitable formula for $r$.

This list of extensions does not include all known applications of the method, but it does indicate that the method of simple correlation has a very broad field of application. The mathematical developments extend far beyond the purposes of this paper.

*Summary.* By simple correlation we seek to measure the strength of common influences which tend to cause two variables to vary in the same direction or in opposite directions. When variation is in the same direction the value of $r$ is positive, and otherwise negative. When a sample is arranged in parallel sub sets the variance $S_m^2$ about the means of sub sets, plus the variance $\sigma_m^2$ of means of the sub sets about the general mean, equals the total variance $\sigma^2$. If $X_1$ is arranged in sub sets which correspond to assigned values of $X_2$ and a straight line is fitted to the sub sets by least squares, it follows that the variance of points about the line, plus the variance of corresponding points on the line, equals the total variance.

Write the relation $S^2 + \sigma_e^2 = \sigma^2$, and Pearson's $r$ is defined as $r = \pm\frac{\sigma_e}{\sigma}$. Regardless of the manner in which values of $X_1$ are classified, the relation $S_m^2 + \sigma_m^2 = \sigma^2$ is true and the ratio $F = \frac{\sigma_m^2}{S_m^2}$ is the basis for Fisher's Analysis of Variance. When $X_1$ is classified according to assigned values of $X_2$ the correlation ratio is defined as $\eta = \pm\frac{\sigma_m}{\sigma}$. The correlation ratio is numerically greater than or equal to $r$. Both $r$ and $\eta$ are numerically less than or equal to one. The value of $r$ is not changed by measuring values of either variable from any assigned origin. The value is not changed if either variable is changed by, adding a constant to each value of the variable, subtracting a constant, multiplying by a constant, or dividing by a constant. If points of the diagram follow a curve more closely than a straight line the measure of non linear correlation has the numerical relation, $r \leqq \rho \leqq \eta$. In cases of large samples a frequency chart may be constructed with consecutive numbers assigned to mid values of the classes. The class numbers are correlated to give the correct result. Crathorne's measures of the correlation ratios follow also from the calculations on the frequency chart. Measures of correlation have been extended to problems involving more than two variables, and to other special problems regarding relations between variables which are classified but not measured. The mathematics of many special problems appears in the literature.

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The meaning of algebra

# A BINOMIAL IDENTITY RELATED TO RHYMING SEQUENCES

Jack Levine

1. *Introduction.* Consider a sequence $a_1a_2\cdots a_n$ of $n$ numbers composed of the $r$ different numbers 1, 2, $\cdots$, $r$, with the number $k$ occurring $n_k$ times, so $n_1+\cdots+n_r = n$. If the first appearance of $k$ in the sequence is in position $p_k$, and if $p_k < p_m$ whenever $k<m$, ($p_1 = 1$), then $a_1a_2\cdots a_n$ is called a rhyming sequence of length $n$.

Let $A(n_1, n_2, \cdots, n_r)$ equal the number of rhyming sequences of length $n = \Sigma n_i$, and containing $n_1$ 1's, $n_2$ 2's, $\cdots$, $n_r$ $r$'s. Robinson [10] proposed the problem of finding the value of the function $A$, and a solution was given by Smith [10]. (An independent solution had been obtained by the writer.) This solution takes the form

$$(1.1)\quad A(n_1,\cdots,n_r)=\binom{n-1}{n_1-1}\binom{n-n_1-1}{n_2-1}\binom{n-n_1-n_2-1}{n_3-1}\cdots\binom{n-n_1-\cdots-n_{r-1}-1}{n_r-1}.$$

(Note the last factor on the right = 1). Thus, the $A(2, 1, 2) = 4$ sequences are 11233, 12133, 12313, 12331.

Now the $A$-numbers are the first of a set of three, the other two being defined by

$$(1.2)\qquad B(n, r) = \sum A(n_1, n_2, \cdots, n_r),$$

$$(1.3)\qquad C(n) = \sum B(n, r).$$

In (1.2) the summation is over all compositions (ordered partitions) $n_1 \cdots n_r$ of $n = \Sigma n_i$ in exactly $r$ parts, so there will be $\binom{n-1}{r-1}$ terms in this sum, MacMahon [8, p. 150]. In (1.3) the sum for $r = 1$ to $n$.

These $B$ and $C$ numbers are well known. $C(n)$ represents the number of rhyming schemes in a stanza of $n$ lines, and $B(n, r)$ is the Stirling number of the second kind. From (1.2) this can also be defined as the number of rhyming schemes of $n$ lines containing exactly $r$ different rhyming lines.

A partial list of references treating these numbers in various forms is given by Becker [1], Bell [2], [3], Browne-Becker [4], Epstein [5], Gupta [6], Mendelsohn-Riordan [9], Whitworth [11, Chap. III]. Further references

are included in these.

Known generating functions for (1.2) and (1.3) are

$$e^{e^x-1} = \sum C(n)\frac{x^n}{n!} \qquad \frac{(e^x-1)^r}{r!} = \sum B(n,r)\frac{x^n}{n!}. \tag{1.4}$$

Gupta [6] has given an elaborate Table of values for $B(n, r)$ and $C(n)$.

Not as well known is the use of these rhyming schemes by cryptographers and cryptanalysts in the classification of words. Here the term pattern sequence is used, every word being classified according to its pattern. Thus, the work MISSISSIPPI has the pattern

M I S S I S S I P P I
1 2 3 3 2 3 3 2 4 4 1,

this pattern being one of the set $A(2, 3, 4, 2)$. Large collections of words arranged according to common pattern have been compiled.

We obtain below a recurrence relation satisfied by the $A$-numbers from which by use of (1.1) the stated binomial identity is derived. Also, a simple proof of the identity of $B(n, r)$ with the Stirling numbers as mentioned above is given.

2. *The binomial identity.* We first give a proof of the recurrence relation

$$\begin{aligned} A(n_1, n_2, \cdots, n_r) &= A(n_1-1, n_2, \cdots, n_r) + A(n_1, n_2-1, n_3, \cdots, n_r) \\ &\quad + \cdots + A(n_1, n_2, \cdots, n_{r-1}, n_r-1), \end{aligned} \tag{2.1}$$

with the convention that if an $n_i - 1 = 0$ on the right side of (2.1), $(i \neq r)$, the corresponding $A$ is set equal to zero, and if $n_r - 1 = 0$, the corresponding $A$ is set equal to $A(n_1, \cdots, n_{r-1})$.

To prove (2.1), first observe that the pattern sequences comprising the set $A(n_1, \cdots, n_r)$ can be split up into $r$ mutually exclusive and exhaustive classes, $G_1(n)$, $G_2(n)$, $\cdots$, $G_r(n)$, where class $G_k(n)$ contains all patterns ending in the number $k$.

Let $G_k'(n-1)$ denote the class of patterns obtained by omitting this final $k$ from the patterns of class $G_k(n)$, and let $T_k(n-1)$ denote the totality of patterns comprising the number $A(n_1, \cdots, n_{k-1}, n_k-1, \cdots, n_r)$. Then the classes $G_k'(n-1)$ and $T_k(n-1)$ are identical. For if not, suppose $x_1x_2\cdots x_{n-1}$ is a pattern in $T_k(n-1)$ but not in $G_k'(n-1)$. Then $x_1\cdots x_{n-1}k$ is in class $G_k(n)$, and as it ends in a $k$, dropping this $k$ must place $x_1\cdots x_{n-1}$ in $G'(n-1)$, a contradiction. The truth of (2.1) now follows.

It is of interest to note that the recurrence relation (2.1) is the same

as that satisfied by the lattice permutation function of MacMahon [8, p. 133].

If now each $A$-number in (2.1) is replaced by its value as given in (1.1) we obtain the identity

$$(2.2)\quad \binom{n-1}{n_1-1}\binom{n-n_1-1}{n_2-1}\cdots\binom{n-n_1-\cdots-n_{r-1}-1}{n_r-1}$$
$$=\binom{n-2}{n_1-2}\binom{n-n_1-1}{n_2-1}\cdots\binom{n-n_1-\cdots-n_{r-1}-1}{n_r-1}$$
$$+\binom{n-2}{n_1-1}\binom{n-n_1-2}{n_2-2}\binom{n-n_1-n_2-1}{n_3-1}\cdots\binom{n-n_1-\cdots-n_{r-1}-1}{n_r-1}$$
$$+\cdots$$
$$+\binom{n-2}{n_1-1}\binom{n-n_1-2}{n_2-1}\cdots\binom{n-n_1-\cdots-n_{r-2}-2}{n_{r-1}-1}\binom{n-n_1-\cdots-n_{r-1}-2}{n_r-2},$$

$(n_1+n_2+\cdots+n_r=n)$.

This identity can also be proved without difficulty directly by induction. If $r=2$, (2.2) reduces to the well-known form

$$\binom{p}{q}=\binom{p-1}{q}+\binom{p-1}{q-1}.$$

(Note that the last factor of each term in (2.2) has the value 1).

From the fact that

$$(1-x)^{-p}=1+R_1^p x+R_2^p x^2+\cdots+R_k^p x^k+\cdots,$$
$$R_k^p=\binom{p+k-1}{p-1},$$

it is easy to see that $A(n_1,\cdots,n_r)$ is the coefficient of $x_1^{n_2+\cdots+n_r}x_2^{n_3+\cdots+n_r}\cdots x_{r-1}^{n_r}$ in the expansion of $(1-x_1)^{-n_1}(1-x_2)^{-n_2}\cdots(1-x_{r-1})^{-n_{r-1}}$.

3. *The numbers* $B(n,r)$. A simple way to show $B(n,r)$ is the Stirling number of the second kind is by means of the recurrence relation

$$(3.1)\qquad B(n,r)=B(n-1,r-1)+rB(n-1,r).$$

To derive (3.1) observe that each $n$-length pattern of $r$ elements can be obtained in one or the other of two ways:

(1) To all $(n-1)$-length patterns of $r-1$ elements add the $r$th element at the

end of each pattern. This gives the term $B(n-1, r-1)$ of (3.1),

(2) To all $(n-1)$-length patterns of $r$ elements add each of the elements $1, 2, \cdots, r$ at the end of the pattern. This gives the term $rB(n-1, r)$.

The difference equation (3.1) is identical with the one satisfied by the stated Stirling's number, and the initial conditions are also the same [7, p. 613]. Becker [1] also derives (3.1) using placements of non-attacking rooks on a triangular chess board.

A second way to obtain $B(n, r)$ depends on the fact there is a 1 - 1 correspondence between the patterns of $B(n, r)$ and the $r$-partitions of $n$ different things. Thus, the correspondence with the $B(4, 2) = 7$ patterns is

| pattern | partition |
|---|---|
| 1112 | (123)(4) |
| 1121 | (124)(3) |
| 1211 | (134)(2) |
| 1222 | (1)(234) |
| 1122 | (12)(34) |
| 1212 | (13)(24) |
| 1221 | (14)(23) |

The second relation of (1.4) then follows by a theorem of Whitworth [11, p. 88].

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# REMARKS ON THE ELEMENTARY SYMMETRIC FUNCTIONS

David Ellis

1. *Introduction.* In a previous paper by this writer[1], it was pointed out that the join operation in a Boolean ring is the sum of the elementary symmetric functions of the joinands. In the present note, the analogous function for the complex field and its extension for Hilbert space is first examined. Secondly, an application of the lattice-theoretic analogs of the elementary symmetric functions is given.

2. *Notation.* If $X_1, \cdots, X_n$ are numbers, $S_{k,n}(X_1, \cdots, X_n)$ shall denote the sum of their products taken $k$ at a time with distinct labels. We set $S_n = \sum_{k=1}^{n} S_{k,n}$. If $X_1, \cdots, X_n$ are under consideration as elements of a lattice, the same notation shall apply, but with arithmetic addition and multiplication replaced with join and meet, respectively.

3. *Expansion of $S_n$.*

*Lemma* 1. If $X_1, \cdots, X_n$ are the zeros of the monic polynomial $f(X)$ over the complex field, then

$$f(X) = X^n + \sum_{r=1}^{n}(-1)^r S_{r,n}(X_1, \cdots, X_n)X^{n-r}.$$

*Proof.* See any standard text on Theory of Equations.

*Corollary* 1. Under the hypotheses of Lemma 1,

$$f(-1) = (-1)^n \left[1 + \sum_{r=1}^{n} S_{r,n}(X_1, \cdots, X_n)\right].$$

*Lemma* 2. Under the hypotheses of Lemma 1,

$$f(X) = \prod_{j=1}^{n} (X - X_j).$$

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(1) David Ellis, On Infinite Series of Sets, Proc., Glasgow Mathematical Assoc., Vol. 2 (1954), pp. 89-92.

*Proof.* See any standard text on College Algebra.

*Corollary* 2. Under the hypotheses of Lemma 1,

$$f(-1) = (-1)^n \prod_{j=1}^{n} (1 + X_j).$$

*Theorem* 1. $1 + S_n(X_1, \cdots, X_n) = \prod_{j=1}^{n} (1 + X_j).$

*Proof.* Combine Corollaries 1 and 2.

*Corollary* 3.

$$S_{n+k}(X_1, \cdots, X_n, \cdots, X_{n+k}) = [1 + S_n(X_1, \cdots, X_n)][1 + S_k(X_{n+1}, \cdots, X_{n+k})] - 1.$$

*Proof.* Apply Theorem 1 to both sides of the equality.

*Corollary* 4.

$$S_{n+k}(X_1, \cdots, X_n, \cdots, X_{n+k}) - S_n(X_1, \cdots, X_n) = S_k(X_{n+1}, \cdots, X_{n+k}[1 + S_n(X_1, \cdots, X_n)].$$

*Proof.* Rearrange Corollary 3.

4. *Connection with Counting.*

*Theorem* 2. If $A_i$ is a set with characteristic function $f_i$ for $i = 1, 2, \cdots, n$, then $\log_2[S_n(f_1(X), \cdots, f_n(X)) + 1] = N(X)$, the number of the sets $A_i$ in which the point $X$ lies.

*Proof.* From Theorem 1,

$$S_n(f_1(X), \cdots, f_n(X)) + 1 = \prod_{j=1}^{n} (1 + f_j(X)) = 2^k \quad \text{where } k = N(X),$$

*Corollary* 5. If $S_n(f_1(X), \cdots, f_n(X)) + 1$ is computed modulo 2, then $N(X)$ as defined in Theorem 2 is a numerical test for symmetric difference points.

5. *Reproductive Property.*

*Lemma* 3. $S_{n_1+n_2}(X_{11}, \cdots, X_{1n_1}, X_{21}, \cdots, X_{2n_2}) =$

$$S_{n_1}(X_{11}, \cdots, X_{1n_1}) + S_{n_2}(X_{21}, \cdots, X_{2n_2}) + S_{n_1}(X_{11}, \cdots, X_{1n_1})S_{n_2}(X_{21}, \cdots, X_{2n_2}) = S_2(S_{n_1}(X_{11}, \cdots, X_{1n_1}), S_{n_2}(X_{21}, \cdots, X_{2n_2}).$$

*Proof.* Direct from change of notation in Corollary 4.

*Theorem* 3.

$$S_t(X_{11}, \cdots, X_{1n_1}, X_{21}, \cdots, X_{2n_2}, \cdots, X_{k1}, \cdots, X_{kn_k})$$

$$= S_k(S_{n_1}(X_{11},\ldots,X_{1n_1}), S_{n_2}(X_{21},\ldots,X_{2n_2}),\ldots, S_{n_k}(X_{k1},\ldots,X_{kn_k}))$$

where $t = \sum_{j=1}^{k} n_j$.

*Proof.* Follows from induction on $k$ utilizing Lemma 3 in both anchor and auxiliary proposition.

6. *Existence of* $S(X_1, X_2, \cdots)$ *in Hilbert Space.*

We write $S(X_1, X_2, \cdots)$ for $\lim_{n\to\infty} S_n(X_1, X_2, \cdots, X_n)$ whenever this limit exists in the broad sense (That is, $S_n$ converges to a point in $[-\infty, \infty]$).

*Lemma* 4. If $\sum_{i=1}^{\infty} X_i^2$ exists in the narrow sense (That is, $\sum_{i=1}^{n} X_i^2$ converges to a point in $(-\infty, \infty)$), then $\lim_{n\to\infty}[\log \prod_{i=1}^{n}(1+X_i) - \sum_{i=1}^{n} X_i]$ exists in the narrow sense.

*Proof.* See page 95 of J. M. Hyslop, *Infinite Series*, Interscience Publishers, Inc., New York, 1950.

*Theorem* 4. If $\sum_{i=1}^{\infty} X_i^2$ converges, then

1). $S(X_1, X_2, \cdots) = -1$ if and only if $\sum_{i=1}^{\infty} X_i = -\infty$.

2). $S(X_1, X_2, \cdots) = a \neq -1$ if and only if $\sum_{i=1}^{\infty} X_i$ converges.

3). $S(X_1, X_2, \cdots) = \infty$ if and only if $\sum_{i=1}^{n} X_i = \infty$.

4). $S_n(X_1, \cdots X_n)$ oscillates if and only if $\sum_{i=1}^{n} X_i$ oscillates.

*Proof.* Theorem 1 and Lemma 4.

*Corollary* 6. $S(X_1, X_2, \cdots)$ is defined in the broad sense for all non-oscillatory points in Hilbert space and, where defined, $S(X_1, X_2, \cdots) > -\infty$.

*Corollary* 7. If $X_1, X_2, \cdots$ is represented by a step-function $f(t)$ on $[1, \infty)$, if $\int_1^{\infty} f^2(t)\,dt$ converges, and $\int_1^{\infty} f(t)\,dt$ converges in the broad sense, then $S(X_1, X_2, \cdots) = \exp(\int_1^{\infty} \ln(1+f(t)\,dt) - 1$.

*Proof.* Theorem 1 and Theorem 4.

7. *Finite Chain Condition in Distributive Lattices.*

For the remainder of this note, the context will be lattice-theoretic and distributivity will be assumed.

*Lemma* 5. For $2 \leqq j \leqq k$,

$$S_{j,k+1}(X_1, \cdots, X_k, X_{k+1}) = (S_{j-1,k}(X_1, \cdots, X_k) \wedge X_{k+1}) \vee S_{j,k}(X_1, \cdots, X_k).$$

*Proof.* Distributivity and definition.

*Theorem* 5. If $a_1, \cdots, a_n$ are elements of a distributive lattice and if $i_1, i_2, \cdots, i_n$ is some permutation of $1, 2, \cdots, n$ so that $a_{i_1} \geqq a_{i_2} \geqq \cdots \geqq a_{i_n}$, then

$$a_{i_j} = S_{j,n}(a_1, \cdots, a_n) \qquad \text{for } j = 1, 2, \cdots, n.$$

*Proof.* Follows from induction utilizing Lemma 5.

*Lemma* 6. For any $X_1, \cdots, X_k$ and $1 \leqq i \leqq j \leqq k$, $S_{i,k}(X_1, \cdots, X_k) \geqq S_{j,k}(X_i, \cdots, X_k)$.

*Proof.* Every joinand on the right precedes some joinand on the left.

*Corollary* 8. For any $X_1, \cdots, X_k$, the points $S_{j,k}(X_1, \cdots, X_k)$ form a chain. For $A$ an infinite subset of the lattice under consideration, let $A^s = \bigcup_{F \subset A} F^s$ where $F$ is a finite set and $F^s = \{S_{1,k}(X_1, \cdots, X_k), \cdots, S_{k,k}(X_1, \cdots, X_k)\}$, where $F = \{X_1, \cdots, X_k\}$.

*Theorem* 6. For a finite set $F$ in a distributive lattice to be a chain it is necessary and sufficient that $F = F^s$.

*Proof.* Lemma 5 and Corollary 8.

*Lemma* 7. If $A$ is any chain in a distributive lattice, then $A^s = A$.

*Proof.* Theorem 6 and heredity.

*Theorem* 7. The condition $A = A^s$ is necessary but not sufficient for an infinite $A$ to be a chain.

*Proof.* Lemma 7 gives necessity. To show lack of sufficiency take $A$ to be any infinite non-linear distributive lattice.

*Theorem* 8. A necessary and sufficient condition that a subset $A$ of a distributive lattice be a chain is that for $X, Y \in A$, $\{X, Y\} = \{X, Y\}^s$.

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# FRENCH GEOMETERS OF THE 19th CENTURY

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(Translated by Richard Villanueva)

Modern geometry came towards the end of the 18th century to contribute in large measure to the renovation of the whole science of mathematics by offering a new and productive approach to research, and above all by showing us with brilliant successes that even in the simplest subject there is much that can be done by an ingenious and inventive mind. The beautiful geometric demonstrations of Huyghens, of Newton, and of Clairaut were forgotten or neglected. The general ideas introduced by Desargues and Pascal remained undeveloped and seemed to have fallen on sterile soil. Carnot by the *Essay on Transversals* and his *Geometry of Position,* and especially Monge, by the creation of *Descriptive Geometry* and by his beautiful theories on the generation of surfaces, came to re-tie a chain which had seemed broken. Thanks to them, the conceptions of the inventors of analytic geometry, Descartes and Fermat, recovered their place beside the infinitesimal calculus of Leibniz and Newton, a place which they had been allowed to lose but would never have to give up again. With his geometry, said Lagrange of Monge, that devil of a man will make himself immortal. And indeed, not only did descriptive geometry allow the coordination and perfection of the methods employed in all the arts, "in which precision of form is a condition of success and excellence for the work and its products;" but it seemed like the graphic translation of a general and purely rational geometry, whose fortunate fertility numerous important researches had demonstrated. Beside *Descriptive Geometry* must be placed that other masterpiece called the *Application of Analysis to Geometry;* nor should we forget that to Monge are attributed the idea of lines of curvature and the elegant integration of the differential equation of these lines for the case of the ellipsoid, which Lagrange, it is said, envied. This feature of all the works of Monge must be stressed. The renovator of modern geometry showed us, from the beginning, that the alliance of geometry and analysis is useful and fruitful, that this alliance is perhaps a condition of success for both.

At the school of Monge, numerous geometers developed. Among them, Poncelet is of the first rank. Neglecting everything in the works of Monge pertaining to the analysis of Descartes or preserving infinitesimal geometry, he concerned himself exclusively with developing the germs contained

in the purely geometric researches of his illustrious predecessor. Made prisoner by the Russians in 1812, on the passage of the Dnieper and confined at Saratov, Poncelet used the spare time which his captivity allowed him to demonstrate the principles which he developed in his *Treatise on the Projective Properties of Figures,* published in 1822, and in his great dissertations on reciprocal polars and on harmonic means which originated at almost the same time. Therefore it was at Saratov, that modern geometry was born.

Poncelet introduced homology and reciprocal polars, thus demonstrating, from the beginning, the fertile ideas from which science evolved for 50 years.

Presented in contrast to analytic geometry, the methods of Poncelet were not received favorably by French analysts. But such were their importance and novelty that they were not long in stimulating the most extensive investigations.

At that time, Gergonne brilliantly edited a periodical which is today invaluable for the history of geometry. *The Annals of Mathematics*, published at Nîmes from 1810 to 1831, was for more than fifteen years the only publication in the entire world devoted exclusively to mathematical research. He was struck by the originality and scope of the discoveries of Poncelet. Already some simple methods of transformation of figures were known; homology had even been used in a plane surface, but without being extended into space, and especially without its power and productiveness being known. Moreover, all these transformations were *exact,* that is, they had one point correspond to one point. By introducing reciprocal polars, Poncelet showed the highest degree of inventiveness; for he gave the first example of a transformation in which a point corresponded to something other than a point. Every method of transformation allows the multiplication of the number of theorems, but that of reciprocal polars had the advantage of having one proposition correspond to another, apparently quite a different, proposition. That was essentially a new achievement. In order to demonstrate it, Gergonne invented the system which has since had such success, that of dissertations written in double columns, with correlative propositions opposite each other, and he had the idea of substituting for the demonstrations of Poncelet, which required as intermediary a curve or a surface of the second order, the famous *principle of duality,* whose significance, a little vague at first, was sufficiently clarified by the discussions of Gergonne, Poncelet, and Plücker.

More recent than Poncelet, who incidentally abandonned geometry for mechanics in which his works had a preponderant influence, Chasles, for whom a chair of *higher geometry* was created in 1947 in the Faculty of Sciences of Paris, endeavored to establish an entirely independent and autonomous geometrical doctrine. He set it forth in two works of high

importance, the *Treatise on Higher Geometry,* published in 1852, and the *Treatise on Conic Sections,* unfortunately unfinished and of which only the first part appeared in 1865.

In the preface of the first of these works, he pointed out very clearly the three fundamental points which permitted the new doctrine to participate in the advantages of analysis and seemed to him to denote scientific progress. They are:

1° – The introduction of the principle of signs, which simplifies both statements and demonstrations, and gives Carnot's analysis of transversals all the scope of which it is capable.

2° – The introduction of imaginaries, which supplements the principle of continuity and supplies demonstrations as general as those of analytic geometry.

3° – The simultaneous demonstration of propositions which are correlative, that is, which correspond by virtue of the principle of duality.

The introduction of the principle of signs was not so novel as Chasles thought at the moment that he wrote his *Treatise on Higher Geometry.* Already Möbius in his *Barycentric Calculus* had carried out a *desideratum* of Carnot, and used signs in the broadest and most precise manner, in defining for the first time the sign of a segment and even of an area.

The second feature which Chasles assigns to his system of geometry is the use of imaginaries. Here his method is really new and he knew how to illustrate it by examples of high interest. Men will always admire the beautiful theories he left us on homofocal surfaces of the second degree, in which all the known properties and other new ones, as varied as they are elegant, derive from the general principle that they are inscribed in a similarly developable surface.

It is, above all, owing to the French scholars Monge, Poncelet, Chasles, and Steiner, "the greatest geometer since Apollonius," that the 19th century was able to know such a thriving in the development of the methods of geometry, the indisputable source of all human knowledge.

## *BIOGRAPHIES*

**Monge** – Gaspard Monge, the count of Péluse, was born at Beaune on May 10, 1746. The son of a peddler, he studied at the college of the Oratoriens. A map he had made of the city of Beaune fell into the hands of an officer who had him enter the School of Mézières, intended to develop engineering officers. In 1768, Monge was named instructor at that school, it being understood that his teaching of descriptive geometry should remain secret and limited to officers of a certain rank.

In 1780, he was designated for a chair of mathematics at Paris. The

first dissertation of importance which he sent to the Academy of Sciences dates from 1781, and has for its subject a discussion of the lines of curvature of a surface.

Monge was an ardent supporter of the doctrines of the Revolution. In 1792 he became Minister of the Navy and helped the Committee of Public Safety by using his science for the defense of the Republic. Denounced when the Terrorists came into power, he escaped the guillotine only by a hasty flight. On his return in 1794, he was named professor at the Normal School, then at the Polytechnic School, where he taught descriptive geometry. His lessons were published in 1800. At the advent of the Restoration, all his duties were taken away and he was deprived of the honors which he had been accorded; this disgrace had a profound effect on his health and he did not survive long. He died in Paris on July 28, 1818.

**Poncelet** – Jean Victor Poncelet, who was born at Metz on July 1, 1788, was an engineering officer. A prisoner of war at the time of the retreat from Moscow in 1812, he used his enforced leisure to review the elements of geometry which he had been previously taught and to investigate some new ideas which this work had suggested to him.

His three remarkable discoveries were set forth in his *Treatise on the Projective Properties of Figures,* published in 1822 and for years the only work properly to initiate mathematicians in that modern geometry which Poncelet has the honor to have founded. He also wrote a treatise on practical mechanics, 1826; a dissertation on water mills and a report on the machines and instruments exhibited by English industry at the London International exposition in 1851. He had numerous articles published in the *Journal de Crelle.* The most remarkable among them gives an explanation, with the help of the principle of continuity, of the imaginary solutions which arise in the solution of geometric problems. Poncelet died in Paris on December 22, 1867.

**Chasles** – Michel Chasles was born at Chartres in 1793 and died in Paris in 1880. A student at the Polytechnic School, he published three geometrical papers whose value was not at all appreciated, and then became a stockbroker's partner, following the wishes of his father. He plunged into mundane life and returned to his first studies only after reverses of fortune. That was in 1827. Ten years later, thanks above all to his *Historical Outline on the Origin and Development of Methods in Geometry* (1834), he had merited and acquired great fame. Chasles did not enter the Academy of Sciences until the age of 58. An illustrious English geometer believed he could say of him that he was "the emperor of geometry."

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Tennie (Sarthe) – France

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Edited by

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# TWO FORMS OF MATHEMATICAL INDUCTION

Arthur Schach

The principle of mathematical induction, as a basis for proving that a theorem or equation $P(n)$ holds for all positive integers $n$, is usually stated in the form:

(I) If $P(1)$ and if, for all $n$, $P(n)$ implies $P(n+1)$, then $P(n)$ for all $n$.

But often it seems necessary, and is certainly convenient, to use the different form:

(II) If $P(1)$ and if, for all $n$, $P(1) \& P(2) \& \cdots \& P(n)$ implies $P(n+1)$, then $P(n)$ for all $n$.

Are (I) and (II) equivalent? Or is it the case, perhaps, that (II)—sometimes called the "strong" form of the principle—implies (I), but that (I) does not imply (II)?[1] The point is not that doubts arise as to the validity of either of the two forms, but that, being so similar, one is curious to know how they might be related.

It would be surprising if this question has not been raised and answered somewhere in the literature. However, it appears not to have been noted in the more obvious places one would look to find it—e.g., systematic works on number systems, mathematical logic, or foundations of mathematics.

At any rate, the question proves not difficult to answer. It can be shown that (1) the two forms of induction are equivalent and (2) any argument based on one of the forms can always be recast so that it is based instead on the other.[2]

A simple proof of the equivalence of (I) and (II) is suggested by current axiomatic developments of the theory of integers. The usual practice is to derive (I), and occasionally (II) as well, from the well-ordering

principle—that every non-empty set of positive integers has a least element.[3] It is then a straightforward matter to construct *reductio ad absurdum* proofs of the converse relations—i.e., to show that each of the induction principles implies the well-ordering principle. Combining these results, one has a proof of the equivalence of (I) and (II).

The indicated proof—which makes an interesting students' exercise in connection with axiomatic treatments of the integers—does of course show that the two forms of induction are equivalent. But it also conceals the fact that the equivalence of (I) and (II) is already assured by their logical form and does not depend on their relation to the well-ordering principle or any other mathematical proposition. Actually, the problem could be more appropriately assigned as an exercise in applying the propositional calculus.[4] A proof of this kind[5] is given below, based on three of the simplest theorems of that calculus. Letting $A$, $B$, and $C$ stand for any propositions whatever, the theorems are these:

T1. If $A$ implies $B$, then $A$ implies $A \& B$.
T2. $A \& B$ implies $B$.
T3. If $A$ implies $B$, then $C \& A$ implies $B$.

The proof itself has the usual two parts.

(a) Assuming (I), to prove (II): We wish to show that, supposing
(1) $P(1)$ and
(2) $P(1) \& P(2) \& \cdots \& P(n)$ implies $P(n+1)$ for all $n$
are given, then $P(n)$ for all $n$.
Let $Q(m)$ be an abbreviation for $P(1) \& P(2) \& \cdots \& P(m)$. Then
(3) $Q(1)$ from (1) and the identity of $P(1)$ and $Q(1)$
(4) $Q(n)$ implies $Q(n+1)$ for all $n$, from (2) by T1
(5) $Q(n)$ for all $n$, from (3) and (4) by (I)
(6) $Q(n)$ implies $P(n)$ by T2
(7) $P(n)$ for all $n$, from (5) and (6).

(b) Assuming (II), to prove (I): This is obvious, but the proof is perhaps short enough to set down. Here we show that when
(1) $P(1)$ and
(2) $P(n)$ implies $P(n+1)$ for all $n$
are given, $P(n)$ holds for all $n$.
(3) $P(1) \& P(2) \& \cdots \& P(n)$ implies $P(n+1)$ for all $n$, from (2) by T3
(4) $P(n)$ for all $n$, from (1) and (3) by (II).

The above proof also shows how, if necessary, one can get along with either of the two forms alone. Thus if (II) has been used in demonstrating some result, a proposition $Q(m)$ can be introduced and with its help the

same result obtained by the use of (I).

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[1] J. B. Rosser, p. 399 of *Logic for Mathematicians* (McGraw Hill, 1953), referring to (I) and (II) as "weak" and "strong" induction, respectively, says that "...one can carry out proofs by strong induction in cases where weak induction would be inadequate."

[2] Some other forms in which induction is occasionally stated appear to be easily reducible to the two considered here. Also, no additional problems are raised by generalized forms of the principle, that is, those used to show that $P(n)$ holds for all $n \geq a$, where $a$ can be any integer.

[3] Both derivations are given, for example, in Birkhoff and MacLane's *A Survey of Modern Algebra*, p. 11f (Macmillan, 1941).

[4] S. C. Kleene, p. 193 of *Introduction to Metamathematics* (Van Nostrand, 1952), indicates a derivation of (II) from a set of logical postulates (those for the propositional calculus plus additional postulates for the predicate calculus) supplemented by a set for number theory (essentially those of Peano) among which (I) is included; the well-ordering principle is derived from the same postulates on pp. 189-191.

[5] Strictly, of course, the proof should be carried out by means of the predicate calculus. The theorems of that calculus required here, however, are so simple that little is gained by referring to them explicitly.

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# SOME ELECTRICAL EXAMPLES TO ILLUSTRATE STOKES'S THEOREM

Walter P. Reid

FOREWORD. One may easily illustrate Stokes's theorem by evaluating both surface and line integrals for a suitable function and surface. The function might be chosen somewhat arbitrarily on the basis, perhaps, of convenience in performing the necessary integrations. However, it is in some respects more satisfactory to consider an example in which the function in question has some physical significance. In the present paper Stokes's theorem is applied to a few simple electrical examples. It will be seen that very little previous knowledge of electricity is needed.

INTRODUCTION. The student studying electricity and magnetism is usually taught Coulomb's law, Faraday's law, Ampere's law, and so forth, and then eventually is given Maxwell's equations. These equations summarize very nicely much of what he has learned. However, Maxwell's equations could equally well be taken as a starting point for a study of electricity and magnetism, and that is what will be done in this paper.

Let it be postulated that, for a homogeneous, isotropic medium, the electric field, $\overline{E}$, and the magnetic field, $\overline{H}$, are related by Maxwell's equations:

$$\nabla \times \overline{E} = -\mu \frac{\partial \overline{H}}{\partial t} = -\mu \dot{\overline{H}} \tag{1}$$

$$\nabla \times \overline{H} = \overline{j} + \epsilon \frac{\partial \overline{E}}{\partial t} = \overline{j} + \epsilon \dot{\overline{E}} \tag{2}$$

where $j$ is the current density, and $\mu$ and $\epsilon$ are assumed to be constants. If at a given point the true charge density is $\rho$, and its velocity is $\overline{v}$, then one may write

$$\overline{j} = \rho \overline{v} \tag{3}$$

Also, where Ohm's law applies,

$$\overline{j} = g \overline{E} \tag{4}$$

With this as a background in electricity and magnetism, one may now illustrate Stokes's theorem:

$$\oint \overline{H} \cdot \overline{\tau}\, ds = \iint \overline{n} \cdot \nabla \times \overline{H}\, dA \tag{5}$$

by applying it to some simple problems in the field of electricity. As a matter of convenience, the examples that follow will just use (2) and not (1). Counter parts of examples 4 and 5 below could easily be used to illustrate (1) if desired.

EXAMPLE 1. STEADY CURRENT IN AN INFINITE STRAIGHT WIRE. Use cylindrical coordinates $r, \phi, z$, with the wire along the $z$ axis, and current flowing in the positive $z$ direction. Apply Stokes's theorem and (2) to the circular area $r \leqq R$ in any plane $z =$ constant. Then $\overline{H} \cdot \overline{\tau} = H_{\phi}$. By symmetry considerations, this is a constant along the circular path, hence

$$\oint \overline{H} \cdot \overline{\tau} ds = H_{\phi} \oint ds = 2\pi R H_{\phi} \tag{6}$$

Since the current is steady, nothing varies with the time, so $\dot{\overline{E}} = 0$ in (2). Also, $\overline{n}$ and $\overline{j}$ are both in the positive $z$ direction, and $\overline{n}$ is a unit vector, so $\overline{n} \cdot \overline{j} = j$ inside the wire, and $\overline{n} \cdot \overline{j} = 0$ outside the wire. Thus the right side of (5) becomes

$$\iint \overline{n} \cdot \nabla \times \overline{H} dA = \iint \overline{n} \cdot \overline{j} dA = \iint j dA \tag{7}$$

Denote the cross-sectional area of the wire by $a$, and the total current by $J$. Assume that the current density in the wire is a constant, or $j = J/a$. Then

$$\iint j dA = j \iint dA = \begin{cases} \pi R^2 J/a & \text{if } R \leqq \text{radius of the wire} \\ J & \text{if } R \geqq \text{radius of the wire} \end{cases} \tag{8}$$

Hence, from (5), (6), (7) and (8):

$$H_{\phi} = RJ/(2a) \quad \text{inside the wire} \tag{9}$$

$$H_{\phi} = J/(2\pi R) \quad \text{outside the wire} \tag{10}$$

EXAMPLE 2. STEADY CURRENT CHARGING A CIRCULAR PARALLEL PLATE CONDENSER. Let the wire be along the $z$ axis, and the plates of the condenser perpendicular to the $z$ axis, with centers on the $z$ axis. The electric field between the plates will first be determined from (10) by applying Stokes's theorem to the surface $F$ shown in Fig. 1. This is a surface of revolution about the $z$ axis, and passes between the plates of the condenser.

The line integral in this case is the same as in Example 1, and so (6) still applies. Since the surface $F$ is nowhere pierced by the wire or a plate of the condenser, $\overline{j}$ is zero everywhere on it. Assume that $\overline{E}$ is in the $z$ direction, and that it is constant throughout the condenser. Then $\overline{n} \cdot \overline{E} = E$ is constant also. Hence

$$\iint \bar{n}\cdot\nabla\times\bar{H}\,dA = \epsilon\iint \bar{n}\cdot\dot{\bar{E}}\,dA = \epsilon\dot{E}\iint dA = \epsilon\dot{E}A \tag{11}$$

The surface integration in (11) is performed over just the part of the surface passing between the plates of the condenser, since elsewhere on the

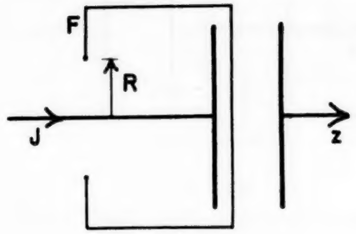


Fig. 1

surface, $E$, and therefore $\dot{E}$, is assumed to be zero. The area $A$ is thus simply the area of a plate of the condenser. From (5), (6), (10) and (11), one obtains

$$\epsilon\dot{E} = J/A \tag{12}$$

If this is integrated with respect to the time, with time taken to be zero when the charge $Q$ on the plates is zero, the electric field between the plates is found to be given by

$$\epsilon E = Q/A \tag{13}$$

Now that the electric field inside the condenser is known, the magnetic field there may be obtained by applying Stokes's theorem to a circular area $r \leq R$ lying between the plates of the condenser and parallel to each plate, with $R$ less than the radius of a plate. For this case, the previous discussion and equations still apply provided that $A$ in (11) is replaced by $\pi R^2$. From (5), (6), and (11) with this change, $H_\phi$ inside the condenser is found to be $\epsilon R\dot{E}/2$. But $\epsilon\dot{E}$ is given by (12), and so the magnetic field inside the charging condenser is

$$H_\phi = R\,J/(2A) \tag{14}$$

Thus the magnetic field inside the condenser increases linearly with $R$, just as it does inside the wire.

It is of interest to note that the result (14) may also be obtained by applying Stokes's theorem to surface $M$ or $P$ of Fig. 2. Surface $P$ has current $J$ passing through it along the wire, and current $(1-\pi R^2/A)J$ passing through it in the other direction along the plates. The difference between these currents is $\pi R^2 J/A$. When this is set equal to expression (6), one

again obtains the result given in (14), this time without the need for knowing the electric field inside the condenser. Surface $M$ is of interest

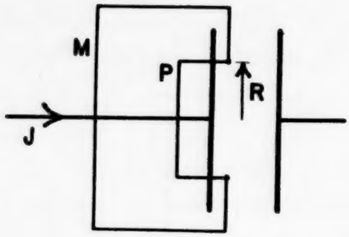


Fig. 2

because over the part of it pierced by the wire, $\nabla\times\overline{H} = \overline{j}$, while over the part of the surface between the plates of the condenser, $\nabla\times\overline{H} = \epsilon\dot{\overline{E}}$. Hence the right side of (5) is in this case equal to $J-\epsilon\dot{E}(A-\pi R^2)$. By using expression (12) for $\epsilon\dot{E}$ one obtains once again the result (14) for the magnetic field inside the condenser.

Throughout this example it has been assumed that the electric field does not vary with position inside the condenser, and stops abruptly at the edge of the condenser. These assumptions are convenient, and approximately true. However, by the use of Stokes's theorem it is easy to show that the electric field actually cannot be zero for $R$ greater than the radius of the plates. It is merely necessary to consider a small rectangle with one edge inside the condenser and parallel to the field, and the opposite side of the rectangle outside the condenser. For the line integral of $\overline{H}\cdot\overline{\tau}$ to be zero for this rectangle as required by Stokes's theorem and (2), it is necessary to have a non-zero field outside the condenser. However, this field falls off rapidly with distance, and is customarily ignored.

EXAMPLE 3. CHARGED DIELECTRIC CYLINDER ROTATING WITH CONSTANT ANGULAR VELOCITY. Consider an infinite right-circular cylinder of radius $a$ with axis along the $z$ axis. Assume that it has a uniform charge density, $\rho$, throughout its volume, and that it is rotating with constant angular velocity, $\omega$.

First apply Stokes's theorem to a rectangle outside the cylinder, lying in a plane $\phi=$constant, and with edges parallel and perpendicular to the $z$ axis. $ABCD$of Fig. 3 is such a rectangle. Since the angular velocity of the cylinder is a constant, nothing varies with the time, and so $\dot{\overline{E}}$ of Eq. (2) is zero everywhere. Also, $\overline{j}$ is given by (3), and so is zero outside the cylinder. Thus $\nabla\times\overline{H}$ is zero everywhere outside the cylinder, and so by Stokes's theorem, the line integral of $\overline{H}\cdot\overline{\tau}$ must be zero also. Therefore $E_z$ must be the same along $DC$ and $AB$, and hence is a constant everywhere outside the rotating cylinder. Take this constant value to be zero.

Next apply Stokes's theorem to a similar rectangle $FGKM$ which has

one edge $FG$ parallel to the $z$ axis and inside the cylinder, and the opposite edge $MK$ outside. The only contribution to the line integral now comes

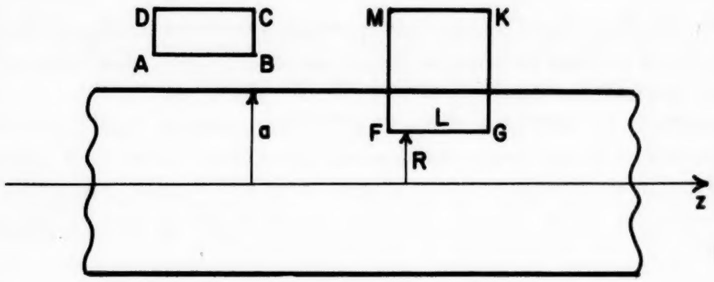


Fig. 3

along $FG$, and since $\overline{H}$ does not vary with $z$, the line integral along $FG$ is simply $H_z$ times the length $L$ of the path, or

$$\oint \overline{H} \cdot \overline{\tau} ds = \int_F^G H_z ds = LH_z \tag{15}$$

By using (2), (3), and $v = r\omega$, the surface integral of Stokes's theorem is found to be

$$\iint \overline{n} \cdot \nabla \times \overline{H} dA = \iint \overline{n} \cdot \overline{j} dA = \int_R^a \int_0^L \rho r\omega dz\, dr = \rho\omega L(a^2 - R^2)/2 \tag{16}$$

The $z$ component of the magnetic field, $H_z$, inside the cylinder is then obtained by equating (15) and (16).

EXAMPLE 4. ELECTROMAGNETIC WAVE ALONG THE INSIDE OF A PERFECTLY CONDUCTING CIRCULAR CYLINDER. Use cylindrical coordinates $r, \phi, z$ with the cylinder given by $r = a$. From Ref. [1] or [2], the components of the electric field for a particular type of wave known as the $E_0$ wave are (with notation changed):

$$\begin{aligned} E_z &= G\gamma J_0(\gamma r) \cos(kz - \omega t), \quad E_\phi = 0, \\ E_r &= GkJ_1(\gamma r) \sin(kz - \omega t) \end{aligned} \tag{17}$$

Apply Stokes's theorem to a circular disk $r \leqq R \leqq a$. Since there is no variation with $\phi$, the line integral of Stokes's theorem is given by (6). In the surface integral, $\overline{j} = 0$, so

$$\begin{aligned} \iint \overline{n} \cdot \nabla \times \overline{H} dA &= \epsilon \iint \overline{n} \cdot \dot{\overline{E}} dA = \epsilon \iint \dot{E}_z dA \\ &= \epsilon\omega\gamma G \sin(kz - \omega t) \int_0^R \int_0^{2\pi} r J_0(\gamma r)\, d\phi\, dr \end{aligned}$$

$$= 2\pi\epsilon\omega G \sin(kz-\omega t) R J_1(\gamma R) \tag{18}$$

The magnetic field, $H_\phi$, is obtained by setting this result equal to expression (6). The $H_\phi$ obtained in this way differs from the $F_\phi$ of Ref. [1] by a factor of $c$. This is because m.k.s. units have been used here, whereas h.l.u. are used in [1].

EXAMPLE 5. OSCILLATING ELECTRIC DIPOLE. Let there be an electric dipole at the origin oscillating along the $z$ axis. Use spherical coordinates $r, \phi, \theta$, where $\theta$ is the angle with the $z$ axis. From Ref. [3], the components of the electric field are found to be the real or imaginary parts of

$$\begin{aligned} E_r &= 2G\left[\frac{i}{(kr)^3} - \frac{1}{(kr)^2}\right]\cos\theta\, e^{i(\omega t-kr)}, \quad E_\phi = 0, \\ E_\theta &= G\left[\frac{i}{(kr)^3} - \frac{1}{(kr)^2} - \frac{i}{kr}\right]\sin\theta\, e^{i(\omega t-kr)} \end{aligned} \tag{19}$$

where $i=\sqrt{-1}$. Apply Stokes's theorem to the spherical cap $r=R$, $\theta \leqq \theta_0$. Then, since there is no variation with $\phi$, the line integral of Stokes's theorem is

$$\oint \overline{H}\cdot\overline{\tau}\,ds = H_\phi \oint ds = 2\pi R H_\phi \sin\theta_0 \tag{20}$$

Also, in (2) $\overline{j}=0$ everywhere on the spherical cap, and $\dot{\overline{E}} = i\omega\overline{E}$. So, for the surface integral of Stokes's theorem one obtains:

$$\begin{aligned} \iint \overline{n}\cdot\nabla\times\overline{H}\,dA &= \epsilon\iint \overline{n}\cdot\dot{\overline{E}}\,dA = \epsilon i\omega \iint \overline{n}\cdot\overline{E}\,dA = \epsilon i\omega \iint E_r\,dA \\ &= 2i\epsilon\omega G\left[\frac{i}{(kR)^3} - \frac{1}{(kR)^2}\right] e^{i(\omega t-kR)} \iint \cos\theta\,dA \\ &= 2\pi i\epsilon\omega R^2 G\left[\frac{i}{(kR)^3} - \frac{1}{(kR)^2}\right] e^{i(\omega t-kR)}\sin^2\theta_0 \end{aligned} \tag{21}$$

The magnetic field $H_\phi$ is determined by equating (20) and (21).

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# A MECHANICAL MODEL WHICH APPROXIMATES THE SUM OF AN ANNUITY*

Roger Osborn

A mechanical device which will approximate the sum of an annuity may be constructed from a typewriter ribbon spool and several yards of ordinary ribbon (the typewriter ribbon being discarded because of the ink). The ribbon should be wound on the spool under relatively constant tension. It would be possible, of course, to devise means by which the physical properties of the model could be improved, but as a device illustrating the snowballing property of the sum of an annuity, the physical properties just described are satisfactory.

Let each complete revolution of the spool be considered as a payment period of an annuity. The hub of the spool used to give the following data is one inch in diameter and hence the length of the first revolution is approximately 3 1/7 inches. Let the length of ribbon wound on the spool be looked upon as the sum of a case I (or simple) annuity. The following partial table of comparative values (correct to three significant digits) may then be constructed. (The results shown are for a particular ribbon.)

| $n$ revolutions or payments | measured length of ribbon (inches) | $3\frac{1}{7}s_{\overline{n}\rceil 1\%}$ | $3\frac{1}{7}s_{\overline{n}\rceil 1\frac{1}{8}\%}$ | $3\frac{1}{7}s_{\overline{n}\rceil 1\frac{1}{3}\%}$ |
|---|---|---|---|---|
| 15 | 51.1 | 50.6 | 50.9 | 52.1 |
| 30 | 112 | 109 | 111 | 115 |
| 45 | 183 | 178 | 183 | 192 |
| 60 | 264 | 257 | 267 | 290 |
| 75 | 355 | 349 | 367 | 401 |

In comparing these values, it appears that the length of ribbon represents the sum of an annuity for an interest rate between 1% and 1 1/8 %. The third rate, 1 1/3 %, appears to be a little too large, but it is included for a reason which will appear shortly. While making comparisons, it should be noticed that it would be more proper for the length of ribbon to be compared with an annuity which is paid continuously and in which the interest

*Portions of this paper were presented at the meeting of the Texas Section of the Mathematical Association of America, April 19, 1958.

rate is compounded continuously, but this comparison differs very little from those given.

Now consider briefly the spiral formed by the ribbon on the spool. It is a spiral of the form $\rho = \frac{k\theta}{2\pi}$, in which $k$ is the thickness of the ribbon. (It happens that the particular ribbon used to give the above data is approximately 1/150 inch thick.) If the radius vector at the beginning of the first winding is $r$, the length of $n$ windings of a ribbon of thickness $k$ wound on such a spool is given by

$$L = \int_{\frac{2\pi r}{k}}^{\frac{2\pi r}{k} + 2\pi n} \left[\frac{k}{2\pi}\sqrt{\theta^2+1}\right] d\theta.$$

For a very thin ribbon the domain of $\theta$ is such that $\sqrt{\theta^2+1} \doteq \theta$. Hence

$$L \doteq \frac{k}{2\pi}\int_{\frac{2\pi r}{k}}^{\frac{2\pi r}{k} + 2\pi n} \theta \, d\theta = 2\pi rn + \pi n^2 k$$

Using the length of the first winding (approximately a circle of radius $r$) as the periodic payment, the sum of $n$ payments of a case I ordinary annuity (at rate $i$ per period) is

$$S = 2\pi r \, s_{\overline{n}|i} = 2\pi r \frac{(1+i)^n - 1}{i}.$$

Expanding $(1+i)^n$ by the binomial theorem, and using the first two terms of the simplified result (an approximation valid only for small values of $n$),

$$S \doteq 2\pi r\left[n + \frac{n(n-1)}{2} i\right] \doteq 2\pi rn + \pi n^2 ri.$$

The first terms of the final expressions for $L$ and $S$, respectively, are identical, but the last terms correspond only if $ri = k$. This can be achieved for many values of $r$ and $i$, but for the spool used to obtain the above data $r = \frac{1}{2}$ inch, and hence $i = 2k$. For $k = 1/150$ inch, the approximate thickness of the ribbon used, $i = 1 1/3\%$. This is the reason for including the rate $1 1/3\%$ in the table above. It can be seen that for a rate this large the device leads to a poor approximation of the sum of the annuity, the percentage error for $n = 75$ being about 12%. For $n < 50$, though, the error does not exceed 5%.

If it were possible to secure a ribbon .01 inches in thickness, and if this ribbon were wound on a spool for which $r = \frac{1}{2}$ inch, then $i$ would have the value 2%. For this rate, the following values of $L$ and $S$ may be found

(the values of $L$ in this example being computed rather than measured).

| $n$ | $L$ | $S$ | actual error $S-L$ | absolute percent error (approximate) |
|---|---|---|---|---|
| 6 | 20.0 | 19.8 | -.2 | 1% |
| 12 | 42.1 | 42.1 | 0 | 0% |
| 24 | 93.7 | 95.5 | 1.8 | 2% |
| 48 | 223 | 249 | 26 | 10% |
| 96 | 591 | 896 | 305 | 34% |

Since $L$, as computed, is a very close approximation of the measured length of ribbon, it can be seen that the mechanical device is highly unreliable for large $n$, the error exceeding 5% for $n > 35$.

For rates of interest not exceeding 3%, and for $25 < n < 100$, it may be determined experimentally that the error $E = S - L$ may be approximated by

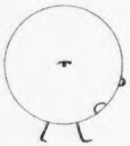
$$E \doteq .604\, i^{5/2} n^{7/2}.$$

This error function should enable an interested reader to determine, for a rate determined by the radius of the hub of a spool and the thickness of ribbon used, the range of values of $n$ for which the mechanical model will give an acceptable approximation of the sum of an annuity.

University of Texas
Austin 12, Texas

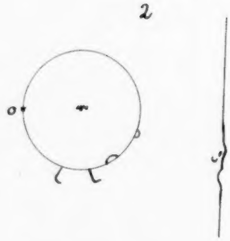
## Mr. Circle

Once upon a time there was created a Circle.



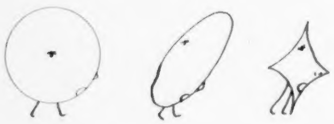
According to Euclid he had an eye in the middle and he was round.

2

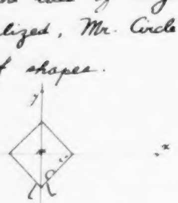


When inversion was discovered, he accidentally passed through the center of inversion, and he was transformed into a tall skinny straight line and his eye jumped to $\infty$.

3



When the idea of length (metric) was generalized, Mr. Circle became all sorts of shapes.





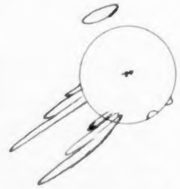
Worse than these, when the example

4

of metric (distance) was:

$$d = |x_1 - x_2| + |y_1 - y_2|,$$

he even looked square.



The worst happend. Non-metric spaces were invented. Then Mr. Circle had to go to Heaven.

A. S. Amer. Mag.

# MISCELLANEOUS NOTES

Edited by

Charles K. Robbins

Articles intended for this department should be sent to *Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Ind.*

## $x^{15} + 1$

Norman Anning

Several generations ago, and somewhere east of the ninetieth meridian, you could buy music lessons on your own Parlor Organ for less than a dollar each. With rates so low the teacher had to be a hustler.

It is reported that a mother, or possibly a grand-mother, asked the natural question: "Oh, Miss Baxter, how is our little Emily really doing?"

She got the reply: "Emily can play *Star of the Sea* and *Little Dog Gone* but she is frightfully weak in her ticknackle."

Instead of attempting to draw a moral let us see what is in the picture. We need some agreements:

1. that, in this paper, $m = -1$,
2. the lemma that

$$x^{15} + 1 = (x+1)(x^2-x+1)(x^4-x^3+x^2-x+1)(x^8+x^7-x^5-x^4-x^3+x+1),$$

and 3. the usual convention that $1 \ m \ 1$ represents the polynomial $x^2-x+1$ and similarly in other cases.

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad m \quad 1 \\
1 \quad 0 \quad 0 \quad 1 \\
1 \quad m \quad 1 \quad m \quad 1 \\
1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \\
1 \quad 2m \quad 3 \quad 3m \quad 3 \quad 2m \quad 1 \\
1 \quad m \quad 1 \quad 0 \quad 0 \quad 1 \quad m \quad 1 \\
1 \quad 1 \quad 0 \quad m \quad m \quad m \quad 0 \quad 1 \quad 1
\end{array}$$

And So On.

The mathematical reader who has mastered as much technique as is

contained in the first four rules will have no trouble in proving the lemma and in showing that the polynomial in each horizontal line is a divisor of $x^{15}+1$. The interested student may be impelled to discover that the "tree" can be extended downward as far as

$$1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1.$$

Let's not waste much sympathy on the boy who has to be propelled to do long division, makes stupid blunders, and says "You gotta show me." He may still lead a rewarding life.

For the student who, by doing the work, has proved that $x^{15}+1$ has polynomial divisors of all degrees from 1 to $(15-1)$, other questions may suggest themselves. Among them, are there integers greater than 15 for which similar statements can be made? How would one prove that the four factors in the lemma are prime factors? They are. But does one need to prove it? Why?

---

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# PAIRING TEAMS

Edmund E. Davis

*Foreword*

The problem is to determine a method of pairing teams, such as in bowling or golf, so that each team will play all the other teams without duplication within a number of weeks which is to be one less than the number of teams meeting, so that any given team can meet each of the other teams only once, and not on the same evening, no team to have to sit out any evening, and no team to be scheduled to any doubleheader on any evening. Random choice establishment of the schedule is not permitted.

The following is an original solution to this problem, and it demonstrates that there is an underlying method which may be used for any number of teams.

Arrange the numbers 1 to $n$ in numerical order in row 1. In the arrangement shown $n = 12$. Enter 1 in each of the rows 2 to $n$ making a diagonal of ones as shown.

| ROWS | TEAMS | | | | | | | | | | | | EVENINGS |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | |
| 2 | | 1 | 11 | 10 | 9 | 8 | 12 | | | | | | 1 |
| 3 | | 12 | 1 | 11 | 10 | 9 | 8 | | | | | | 2 |
| 4 | | | 2 | 1 | 11 | 10 | 9 | 12 | | | | | 3 |
| 5 | | | 12 | 2 | 1 | 11 | 10 | 9 | | | | | 4 |
| 6 | | | | 3 | 2 | 1 | 11 | 10 | 12 | | | | 5 |
| 7 | | | | 12 | 3 | 2 | 1 | 11 | 10 | | | | 6 |
| 8 | | | | | 4 | 3 | 2 | 1 | 11 | 12 | | | 7 |
| 9 | | | | | 12 | 4 | 3 | 2 | 1 | 11 | | | 8 |
| 10 | | | | | | 5 | 4 | 3 | 2 | 1 | 12 | | 9 |
| 11 | | | | | | 12 | 5 | 4 | 3 | 2 | 1 | | 10 |
| 12 | | | | | | | 6 | 5 | 4 | 3 | 2 | 1 | 11 |

At the right of each 1 in rows 2 to $n$ enter the numbers $n-1$, $n-2$, etc. ($n-1 = 11$, $n-2 = 10$, etc.) wherever the number entered is *larger than* the number directly above it in row 1.

At the left of each 1 in rows 2 to $n$ enter the numbers 2, 3, 4, $\cdots$ (from right to left) wherever the number entered is *smaller than* the number directly above it in row 1.

Complete the arrangement by placing $n$ $(=12)$ in each of the rows 2 to $n-1$ $(=11)$. Place it $(n)$ at the right of the numbers previously entered in the even numbered rows and at the left of the numbers already entered in the odd numbered rows. These "$n$" entries are circled.

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Wilmington 4, Delaware

# A CLASSIC ROADBLOCK IN EFFORTS TO PROVE FERMAT'S LAST THEOREM

Glenn James

Attempts to prove Fermat's Last Theorem are always before us. We like to point out the troubles in them in order to encourage the authors. But this is a time consuming job. So we are going to try to save future authors and ourselves a lot of time and energy by discussing here the most prevalent difficulty with these attempts.

As you no doubt know, Fermat's Last Theorem is as follows:

The equation

$$x^n + y^n = z^n \tag{1}$$

has no solution in integers, $x$, $y$, $z$, $n\,(n > 2)$.

Disregarding the fallacy with which this note is concerned, we will first prove that F.L.T. is false and then prove that it is true.

The theorem is false:

Proof. Substitute $x = u^{2/n}$, $y = v^{2/n}$, $z = w^{2/n}$. We obtain

$$u^2 + v^2 = w^2.$$

This Pythagorean equation has, of course, infinitely many integral solutions, $u$, $v$, $w$. Therefore (1) has plenty of integral solutions. However, it remains to prove that the transformation we have used carries at least one set of these $u$, $v$, $w$ into integral $x$, $y$, $z$.

The theorem is true:

Proof. Since it is well known that (1) has no integral solution for $n = 3$, we substitute $x = u^{3/n}$, $y = v^{3/n}$ and $z = w^{3/n}$ in (1) obtaining

$$u^3 + v^3 = w^3$$

This equation has no solution in integers. Hence (1) has none. "Q.E.D."

More generally, suppose we make the transformation

$$x = f_1(u, v, w), \quad y = f_2(u, v, w), \quad z = f_3(u, v, w) \tag{2}$$

in equation (1) and by a proper choice of $f_1$, $f_2$, $f_3$ are able to show that

$$f_1^{\,n} + f_2^{\,n} = f_3^{\,n} \tag{3}$$

has no solution for some set of values of $u$, $v$, $w$ (not necessarily integers). Then it remains only to show that these values substituted in (2) give all possible sets of integers, $x$, $y$, $z$. Indeed this step should be ingated before spending a lot of time juggling equation (3).

---

Editor

## SHOULD YOUR CHILD BE A MATHEMATICIAN?

"Someone surely is going to work out the mathematical formulas that will enable mankind to travel safely to Mars and far beyond into distant space.

"It could be your son or daughter."

Solving the mysteries of outer space is just one of the challenging assignments that await the young man or woman who chooses a career in mathematics, says Norris E. Sheppard, professor of mathematics at the University of Toronto.

"Recognition among science's immortals" can be the reward for the genuinely gifted child who enters the mathematics field, Dr. Sheppard adds. Dr. Sheppard discusses mathematics as a career in an article, "Should Your Child Be A Mathematician?", currently appearing nationally as a public service advertisement of the New York Life Insurance Company.

Dr. Sheppard warns, however, that "to get ahead in mathematics, a youngster must be good." He agrees with a noted mathematician who remarked recently: "Among mathematicians, there is no place for the so-called average man."

Professor Sheppard urges parents to weigh carefully whether a child has the proper qualifications for the field. "Does he have a keen, logical mind and an insatiable curiosity? Is he imaginative? Does he relish his mathematics courses and earn top grades in them? Is he quick at solving mathematical problems in his head? The answer to all these should be a resounding 'Yes'!"

Mathematics is "more than the science of numbers," Dr. Sheppard says. "It is even more than a science that allows us to grasp the real significance of time and space. It is the science that trains a man to cope with unknown quantities and to translate their relationships into logical, comprehensible patterns."

Professor Sheppard says that when work began on the Nike, the Army's anti-aircraft missile, a team of top experts was gathered from the scientific and military fields. "They were unable to move, however, until the mathematicians outlined the way," he says. "It took mathematicians to spot the essential ideas that lay obscured among the many details and divergent languages of the other sciences," he adds.

Though the mathematician is making his mark in many fields, Professor Sheppard says there is now "a crying need for new people." Fewer than one in 7,000 persons in the United States and Canada, excluding secondary school teachers, are earning a living as professional mathematicians,

he says.

"Universities, industrial companies, insurance firms, other businesses and government agencies are all hungry for mathematically trained personnel," says Dr. Sheppard. Mathematicians are also badly needed, he notes, in the special fields of research, teaching, statistics, applied mathematics and actuarial science.

Of educational preparation for a career in mathematics, Dr. Sheppard says, "the more of it the better. A Ph.D. is now almost imperative, both in industry and the academic world."

Because of the availability of large numbers of scholarships and fellowships, Dr. Sheppard believes there is no good financial reason why the talented student should have to go without the education he needs for success in mathematics.

He points out that salaries and advancement prospects in the field are, in the main, excellent, especially in industry, where an applied mathematician with a Ph.D. can start at $7,200 and reach $30,000 a year. In whatever field the mathematician enters, Professor Sheppard says, he can count on a high degree of security. Pension programs are widespread and personnel turnover is small.

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New York Life Insurance Company

# PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton, Los Angeles City College

**Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.**

**Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction.**

**Send all communications for this department to** *Robert E. Horton, Los Angeles City College, 855 North Vermont Ave., Los Angeles 29, California.*

## PROPOSALS

**355.** *Proposed by P.B. Jordain, New York, New York.*

A wine merchant had a small cask containing 100 gallons of wine. In order to make more money, he decided to replace each gallon he took out of the cask by a gallon of water. This he did $n$ times. Finding that he was losing customers, he naively tried to undo what he had done by selling his watered wine from the same cask, but replacing each gallon sold of the mixture by a gallon of pure wine, feeling that by this method, at the end of an additional $n$ gallons of watered wine sold, he would have a cask of full wine again. He was unfortunate enough in having hit upon a total number of gallons sold such that the wine contained in the cask was at a minimum at the end of this operation. The question is, how many gallons of watered wine had he sold in all? Assume that the merchant dealt only in full gallon sales.

**356.** *Proposed by M.S. Klamkin, AVCO, Lawrence, Massachusetts.*

Determine the shortest distance on the right circular cylinder $r=R$, $z=0$, $z=H$ between the two points $P_1(r_1, \theta_1, 0)$; $P_2(r_2, \theta_2, H)$ and also between the two points $P_3(R, \theta_3, z_3)$ and $P_4(R, \theta_4, z_4)$.

**357.** *Proposed by Joseph Andrushkiw, Seton Hall University, New Jersey.*

Show that if $d > 0$,

$$\sum_{k=0}^{\infty} 1/(k^2 - 2dk + 2d^2) = k + 3\pi/4d, \quad 0 \leq k < 1/2d^2$$

**358.** *Proposed by C.W. Trigg, Los Angeles City College.*

A ball having fallen from rest a vertical distance $h$, strikes a stone

protruding from a wall and bounces off horizontally without spinning. If the distance of the stone from the ground is $s$ and the coefficient of restitution is $e$, show that

a) the ball will strike the ground at a distance $2\sqrt{she}$ from the foot of the wall;

b) the inclination of the stone to the horizontal is arc tan $\sqrt{e}$.

**359.** *Proposed by Norman Anning, Alhambra, California.*

Prove that the string-of-pearls polynomial $x^k+1$ can be expressed in at least one non-trivial way as the sum of two squares, if $k$ is any even positive integer different from 2, 4, 8, 16 ....

**360.** *Proposed by Chih-yi Wang, University of Minnesota.*

Considering the higher differences of $\binom{n}{x}$ with respect to $x$ show that

$$\Delta^m \binom{n}{x} = \frac{1}{(n-x)!}\left| D_t^{n-x}[t^n(t-2)^m]\right|_{t=1}$$

**361.** *Proposed by N. A. Court, University of Oklahoma.*

A variable triangle inscribed in a rectangular hyperbola has a fixed vertex and the opposite side moves parallel to itself. Show that its variable nine-point circle passes through two fixed points.

## SOLUTIONS

**ERRATA.** The name of *P. D. Thomas, Coast and Geodetic Survey, Washington, D.C.* was inadvertantly omitted from among the solvers of Problem 326, May 1958, Vol. 31, No. 5. In the same problem in the third line from the bottom of page 292, the denominator of the second term on the right should read $t^2-t+1$ instead of $t^2-t-1$. In the last term of the formula at the top of page 293 the $\alpha$ and $\beta$ should be interchanged.

### An Irregular Area

**334.** [March 1958] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.*

Find the simplest expression for the area $S$ enclosed by the arc $AM$ of a cycloid, the arc $TM$ of the rolling circle $\Omega(a)$ and the base line segment $AT$.

*Solution by J. W. Clawson, Collegeville, Pennsylvania.* Draw $MN$ and $CT$ perpendicular to $AT$. Let angle $MCT = \theta$ and $CT = a$. The area required = area $AMN$ + area trapezoid $NMCT$ − area sector $MCT$.

Now, for $M$, $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$.

$$\text{Hence area} = a^2\int_0^\theta (1-\cos\theta)^2\,d\theta + \frac{a^2\sin\theta}{2}(2-\cos\theta) - \frac{a^2\theta}{2}$$

$$= 3/2\,a^2\theta - 2a^2\sin\theta + (a^2/2)\sin\theta\cos\theta + a^2\sin\theta - (a^2/2)\sin\theta\cos\theta$$

$$= a^2(\theta - \sin\theta)$$

$$= ax.$$

*Also solved by Stanley P. Franklin, Memphis State University; Joseph D.E. Konhauser, State College, Pennsylvania; Arne Pleijel, Trollhattan, Sweden and the proposer.*

## Harmonic Series

**355.** [March 1958] *Proposed by Robert E. Shafer, University of California Radiation Laboratory.*

Prove

$$\sum_{i=1}^{n} \frac{(-1)^{i+1}}{i}\binom{n}{i} = \sum_{i=1}^{n} \frac{1}{i}$$

for all $n \geqq 1$.

**I.** *Solution by Arne Pleijel, Trollhattan, Sweden.* We have, through the substitution $z = 1-u$, the equivalence

$$\int_0^1 \frac{1-(1-u)^n}{u}\,du = \int_0^1 \frac{1-z^n}{1-z}\,dz$$

or

$$\int_0^1\left[\sum_{i=1}^{n} (-1)^{i+1}\binom{n}{i}u^{i-1}\right]du = \int_0^1\left[\sum_{i=1}^{n} z^{i-1}\right]dz$$

thus

$$\sum_{i=1}^{n} (-1)^{i+1}\binom{n}{i}\frac{1}{i} = \sum_{i=1}^{n} 1/i$$

**II.** *Solution by R.G. Buschman, University of Wichita.*

$$\sum_{i=1}^{n} \frac{1}{i} = \psi(n+1) - \psi(1), \quad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

In the factorial series expansion of $\psi(z)$ which converges for $Re(z+a)>0$

$$\psi(z+a) = \psi(a) + \sum_{i=1}^{\infty} (-1)^{i+1} \frac{1}{i} \frac{z(z-1)\cdots(z-i+1)}{a(a+1)\cdots(a+i-1)},$$

substitute $z = n$ and $a = 1$ and note that the series breaks off yielding

$$\psi(n+1) - \psi(1) = \sum_{i=1}^{n} (-1)^{i+1} \frac{1}{i} \binom{n}{i}.$$

(The formulas used are from HIGHER TRANSCENDENTAL FUNCTIONS, Erdélyi et al, section 1.7, numbers (10), (1), and (30).

*Also solved by M.T.Bird, San Jose State College; Brian Brady, Penrith, Australia; John L. Brown, Jr., Pennsylvania State University; Richard Gabel, Arlington, Virginia; J.M.Gandhi, Thappar Polytechnic and School of Engineering, Patiala, India; Harry M. Gehman, University of Buffalo; Joseph D.E.Konhauser, State College, Pennsylvania; F.D.Parker, University of Alaska; Gideon Peyser, Newark College of Engineering, New Jersey; C.F.Pinzka, University of Cincinnati; Robert J. Wagner, Lebanon Valley College, Pennsylvania; and Dale Woods, Idaho State College.*

*Pinzka* pointed out that the problem appears in the "American Mathematical Monthly" as E864, January 1950. *Brown* noted the problem in AN INTRODUCTION TO PROBABILITY THEORY AND ITS APPLICATIONS by William Feller, 1st edition, p 49.

## Coefficient of Friction

**336.** [March 1958] *Proposed by C.W.Trigg, Los Angeles City College.*

A uniform bar with rounded ends and length $x$ is hung from one end by a string of length $x$ and negligible mass.. The other end of the string is attached to a vertical wall. When the free end of the bar is placed against the wall, it is found that $\theta$ is the smallest angle that can be made between the bar and the wall so that the bar will not slip and fall down. (The plane of the bar and string is perpendicular to the wall.) What is the coefficient of friction between the bar and the wall?

*Solution by C.F.Pinzka, University of Cincinnati.* Let $N$ be the normal force that the wall exerts on the bar. Then $\mu N$, where $\mu$ is the coefficient of friction, is the vertical force of friction when motion impends. The line of action of the force of gravity on the bar intersects the string at a point

a distance $x/2\sin\theta$ from the wall and a distance $3x/2\cos\theta$ above the point of contact. Taking moments about this point of intersection, we have

$$\left(\frac{x}{2}\sin\theta\right)(\mu N) = \left(\frac{3x}{2}\cos\theta\right)(N), \quad \text{or } \mu = 3\cot\theta.$$

*Also solved by William E.F. Appuhn, St. John's University, New York; Dermott A. Breault, Sylvania Electric Products Inc., Waltham, Massachusetts; J.W. Clawson, Collegeville, Pennsylvania; G.I. Gaudry, Mackay High School, Mackay, Australia; Joseph M.C. Hamilton, Los Angeles City College; Grant F. Heck, Lebanon Valley College, Pennsylvania; Joseph D.E. Konhauser, State College, Pennsylvania; and M. Morduchow, Polytechnic Institute of Brooklyn.*

## A Pythagorean Triangle

**337.** [March 1958] *Proposed by Victor Thebault, Tennie, Sarthe, France.*

Determine the right triangles whose sides are integers and whose hypotenuse is twice a square.

*Solution by Leon Bankoff, Los Angeles, California.* Let $x$, $y$ denote the legs, and $z = 2a^2$ the hypotenuse of the required triangles.

By a familiar two-parameter representation,

$$x = 2pq \tag{1}$$

$$y = p^2 - q^2 \tag{2}$$

$$z = 2a^2 = p^2 + q^2 \tag{3}$$

Now, all solutions of (3) are given by Frenicle's set (See Dickson, HISTORY OF THEORY OF NUMBERS, Vol. II, p. 435):

$$p = |g^2 - h^2 + 2gh|$$

$$q = |g^2 - h^2 - 2gh|$$

$$a = (g^2 + h^2)$$

with no restrictions on the integers $g$ and $h$. We may therefore write equations (1), (2), and (3) as

$$x = 2\,|g^2 - h^2 + 2gh|\,|g^2 - h^2 - 2gh| = 2\,|g^4 - 6g^2h^2 + h^4|$$

$$y = |g^2 - h^2 + 2gh|^2 - |g^2 - h^2 - 2gh|^2 = 8gh|g^2 - h^2|$$

$$z = 2a^2 = 2(g^2 + h^2)^2$$

*Also solved by William E.F. Appuhn, St. John's University, Brooklyn;*

*George Bergman, Stuyvesant High School, New York; D.A.Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; L.Carlitz, Duke University; Monte Dernham, San Francisco, California; Herbert R. Leifer, Pittsburgh, Pennsylvania and Dale Woods, Idaho State College.*

## A Vector Field

**338.** [March 1958] *Proposed by M.S.Klamkin, AVCO, Lawrence, Massachusetts.*

The vector field $\frac{R}{r^3}$ satisfies the equations $\nabla\times\frac{R}{r^3} = 0$ and $\nabla\cdot\frac{R}{r^3} = 0$. Consequently, this field has a scalar potential and a vector potential. The scalar potential is well known to be $\frac{1}{r}$. Determine the vector potential.

*Solution by D.A.Breault, Sylvania Electric Products Inc.* Under the given conditions, we are asked to determine $G = G(R)$, such that,

$$f(R) = R/r^3 = \text{curl}\, G(R) \tag{1}$$

Since $f(R)$ is solenoidal, $G(R)$ is given by

$$G(R) = -R\times\int_0^1 f(tR)t\,dt \tag{2}$$

whenever the integral exists. In this case it does exist, and is in fact equal to $1/3\, R\times R/r^3$, whence

$$G(R) = 1/3\, R\times R/r^3 + \nabla\emptyset, \tag{3}$$

where $\emptyset$ is an arbitrary scalar function whose second partials exist, (see Brand, ADVANCED CALCULUS, p. 391 ff.)

## Cantor's Discontinuum

**339.** [March 1958] *Proposed by D.F.Huntington and D.A.Kearns, University of Maine.*

Criticize the following "proof" that the Cantor Ternary Set (Cantor Discontinuum) is non-denumerable.

At the $n$th stage of the construction of the set there are $2^n$ closed intervals, the $2^{n+1}$ end points of which belong to the set. Since there are a denumerable number of stages in the construction, the total number of end points is $2^{\aleph_0+1} = 2^{\aleph_0} = \aleph$. Therefore these end points are non-denumerable, and since they constitute a subset of the Cantor set, the entire

Cantor set is non-denumerable.

**I.** *Comments by Lawrence A. Ringenberg, Eastern Illinois University.* The total number of endpoints is $\aleph$, not $2^{\aleph_0}$. The given "proof" contains the erroneous reasoning that since there are $2^{n+1}$ endpoints at the $n$th stage, there must be $2^{\aleph_0}$ endpoints in the Cantor set.

**II.** *Comment by Joseph D.E. Konhauser, Haller, Raymond and Brown, Inc., State College, Pennsylvania.* The "proof" is at fault in the assertion that the number of end points is $2^{\aleph_0+1}$. The number of end points is given by $2^{\omega+1} = \omega$, where $\omega$ is the order type of the set of positive integers and has cardinal $\aleph_0$. The set of end points is denumerable since it is the union of a denumerable number of finite sets.

**III.** *Comment by the proposers.* Of course the proof is invalid since the end points are denumerable. (Each of them has a finite ternary decimal expansion, for example.) The fallacy lies in identifying $\aleph_0$ as a non-existent "last" finite cardinal rather than the cardinal number of the set of finite cardinals. In other words, it is not true that $\lim_{n\to\infty} n$ is $\aleph_0$.

## The Reimann Zeta-Function

**340.** [March 1958] *Proposed by R.G.Buschman, University of Wishita.* Prove that

$$\sum_{k=2}^{\infty} \frac{(-1)^k (k-1)}{k(k+1)} \zeta(k) = 2\int_0^{\infty} ([e^t]-e^t+1/2)\, dt$$

where $[x]$ represents the greatest integer not exceeding $x$ and $\zeta(k)$ is the Reimann Zeta-function.

*Solution by L.Carlitz, Duke University.*

Let $\psi(y) = y-[y]-1/2$, so that $\psi(y+1) = \psi(y)$. Then we have

$$\int_0^{\infty}([e^y]-e^y+1/2)\,dy = -\int_0^{\infty} \psi(e^y)\,dy$$

$$= -\int_1^{\infty} \frac{\psi(x)}{x}dx = -\sum_{n=1}^{\infty}\int_n^{n+1} \frac{\psi(x)}{x}dx$$

$$= -\sum_{n=1}^{\infty}\int_0^1 \frac{x-1/2}{x+n}dx = -\sum_{n=1}^{\infty}\left\{1-(n+1/2)\log\frac{n+1}{n}\right\}$$

$$= -\sum_{n=1}^{\infty}\left\{1-(n+1/2)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{kn^k}\right\}$$

$$= -\sum_{n=1}^{\infty}\left\{\frac{1}{4n^2}+\sum_{k=3}^{\infty}(-1)^k\frac{n+1/2}{kn^k}\right\}$$

$$= -\frac{1}{4}\zeta(2)-\sum_{k=3}^{\infty}\frac{(-1)^k}{k}\zeta(k-1)-\frac{1}{2}\sum_{k=3}^{\infty}\frac{(-1)^k}{k}\zeta(k)$$

$$= \frac{1}{12}\zeta(2)+\sum_{k=3}^{\infty}\frac{(-1)^k}{k+1}\zeta(k)-1/2\sum_{k=3}^{\infty}\frac{(-1)^k}{k}\zeta(k)$$

$$= 1/2\sum_{k=2}^{\infty}(-1)^k\frac{k-1}{k(k+1)}\zeta(k).$$

*Also solved by the proposer.*

**317.** [September 1957] A comment on the proposer's solution to this problem submitted by *Walter R. Talbot, Lincoln University, Missouri.*

In the proposer's solution the statement appears, "...to prove $(e+x)^{(e-x)} > (e-x)^{(e+x)}$, we have the equivalent statement $(e+x)^{(e-x)} > (e-x)^{(e-x)}$, etc."

Using this approach, I fell into a consideration of the relationships existing between $(e-x)^{(e+x)}$ and $(e-x)^{(e-x)}$. I believe the point is significant because the relative sizes of these two quantities depends upon whether $e-x>1$, $e-x=1$, or $e-x<1$.

It is obvious that if $e-x>1$, then $(e-x)^{(e+x)} > (e-x)^{(e-x)}$.

It is obvious that if $e-x=1$. the two quantities are equal.

If $e-x<1$, then $(e-x)^{(e+x)} < (e-x)^{(e-x)}$ as can be shown by letting $e = 2.7$ and $x=2$. $.7^{4.7} < .7^{.7}$.

## QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

**Q 231.** Prove that $N!$ can not be a perfect square [*Submitted by M. S. Klamkin*]

**Q 232.** Find an integral root of $360 = (x-2)(x-3)(x-4)(x-5)$. [*Submitted by*

*J. M. Howell*]

**Q 233.** Consider the tangents of 117°, 118° and 125°. Prove that their sum is equal to their product. [*Submitted by Norman Anning*]

**Q 234.** If the sum of the coefficients of $A(x)B(x)$ is zero, the sum of the coefficients of one of the polynomials is necessarily zero. [*Submitted by Huseyin Demir*]

**Q 235.** On a 26 question test, 5 points were deducted for each wrong answer and 8 points were credited for each correct answer. If all questions were answered, how many were correct if the score was zero? [*Submitted by C. W. Trigg*]

## ANSWERS

**A 231.** The proof follows from the fact that there is always a prime between $r$ and $2r$ for all $r > 1$.

**A 232.** Noting that $720(x-6)! = 2(x-2)!$, we have $6!(x-6)! = 2!(x-2)!$ or

$$\frac{x!}{6!(x-6)!} = \frac{x!}{2!(x-2)!}$$

so $\binom{x}{6} = \binom{x}{2} = \binom{x}{x-2}$ and $x - 2 = 6$ or $x = 8$.

**A 233.** We have $117° + 118° = 360° 125°$.
Tan $(117° + 118°) = -\tan 125°$. The rest of the solution is routine.

**A 234.** For $x = 1$ we have $\Sigma a_i \cdot \Sigma b_i = \Sigma c_i$ and the result follows.

**A 235.** The number of answers in each category is inversely proportional to the value, so there were $(5/13)(26) = 10$ correct answers.

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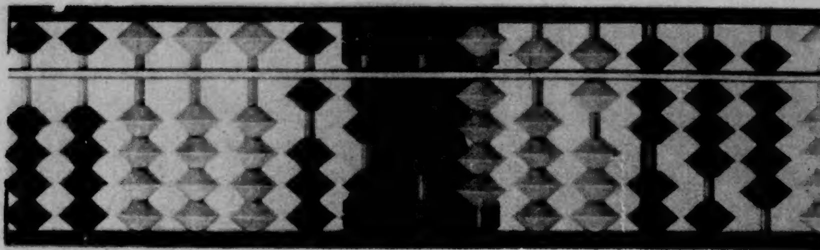

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